

# EXISTENCE OF NONNEGATIVE SOLUTIONS TO STOCHASTIC THIN-FILM EQUATIONS IN TWO SPACE DIMENSIONS

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ABSTRACT. We prove the existence of martingale solutions to stochastic thin-film equations in the physically relevant space dimension  $d = 2$ . Conceptually, we rely on a stochastic Faedo-Galerkin approach using tensor-product linear finite elements in space. Augmenting the physical energy on the approximate level by a curvature term weighted by positive powers of the spatial discretization parameter  $h$ , we combine Itô's formula with inverse estimates and appropriate stopping time arguments to derive stochastic counterparts of the energy and entropy estimates known from the deterministic setting. In the limit  $h \searrow 0$ , we prove our strictly positive finite element solutions to converge towards nonnegative martingale solutions — making use of compactness arguments based on Jakubowski's generalization of Skorokhod's theorem and subtle exhaustion arguments to identify third-order spatial derivatives in the flux terms.

## 1. INTRODUCTION

We are concerned with stochastic thin-film equations of the generic form

$$du = -\operatorname{div} \{m(u)\nabla(\Delta u - F'(u))\} dt + \operatorname{div} \{\sqrt{m(u)}d\mathbf{W}\} \quad (1.1)$$

on a space-time cylinder  $\mathcal{O} \times (0, T]$  where  $\mathcal{O}$  is a bounded rectangular domain in  $\mathbb{R}^2$ . Such kind of equations have been introduced to model dewetting of unstable liquid films under the influence of thermal fluctuations. Here, the mobility  $m(\cdot)$  may be chosen as  $m(u) = u^3 + \beta u^2$ , a prototypical example for the effective interface potential  $F(u)$  is  $F(u) = u^{-8} - u^{-2} + 1$ . It is based on a 6 – 12 Lennard–Jones pair potential and it models disjoining/conjoining van der Waals interactions. Numerical simulations in 1D have shown (see [21, 47]) that discrepancies with respect to time scales of dewetting between physical experiment and deterministic numerical simulation can be overcome if appropriately scaled noise terms are considered. The equations used for those models come along with a number of intrinsic difficulties. First, the degeneracy of the mobility  $m(\cdot)$  in  $u = 0$ , secondly the singular behaviour of the effective interface potential at  $u = 0$ , and thirdly the fact that even in the deterministic case it is still an open problem whether solutions are continuous or bounded if the spatial dimension is at least  $d = 2$ . This is in

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sharp contrast to the one-dimensional case where Sobolev embedding results are the key to establish Hölder continuity in space and time.

The scope of the present work is threefold. First, we wish to establish the existence of martingale solutions for a model problem with quadratic mobility  $m(u) = u^2$ . Secondly, we shall address the case that the stochastic integral in (1.1) is to be understood in the sense of Stratonovich. It turns out that the correction term necessary to rewrite the Stratonovich integral as an Itô integral can be included into the frame of the generic form (1.1) just as a modification of the potential  $F(\cdot)$ . In particular, this modification does not affect the structural conditions formulated on  $F$  in the Itô-case, provided some natural hypotheses on the decay parameters of the corresponding Wiener processes are met.

Growing interest in stochastic thin-film equations with Stratonovich noise arose with the work of Gess and Gnann [31] who proved existence of solutions to stochastic versions of thin-film equations driven only by surface tension, i.e. with  $F(\cdot) \equiv 0$ . Using a novel approximation scheme, Dareiotis, Gess, Gnann, and the second author of this paper [16] extended this result to stochastic thin-film equations with conservative nonlinear multiplicative noise, covering in particular the case of a no-slip condition at the liquid-solid interface which corresponds to  $m(u) = u^3$ . While the results of [31] and of [16] still do not allow for compactly supported initial data, Klein and the second author of this paper use the 1D-predecessor of this work (see [28]) to construct nonnegative solutions to initial data with compact support [46]. In this spirit, the present paper may similarly serve as a starting point to obtain solutions to equations with compactly supported initial data in the 2D setting.

Note that Davidovitch et al. [17] derived stochastic thin-film equations for surface tension driven flow to study numerically the influence of thermal noise on the spreading behaviour. For mathematically rigorous results on the noise impact on free boundary propagation, see [27, 20, 40, 30, 2] and the references therein which are preliminary studies devoted to stochastic second order degenerate parabolic equations of porous-media and of parabolic  $p$ -Laplace-type. However, it is worth mentioning that the techniques of [27, 40] are expected to be universal in the sense that they are based on energy methods developed in [15, 43, 1, 12, 26, 25] which work for general classes of degenerate parabolic equations of second and higher order.

Finally, as the third perspective of the techniques developed in the present paper, we recall that Cahn–Hilliard equations with degenerate mobility are intimately related to thin-film equations [22, 41]. Hence, we expect only slight modifications to be required to obtain existence results for Cahn–Hilliard equations with Stratonovich conservative noise – for results on stochastic degenerate Cahn–Hilliard equations with non-conservative noise, we refer to the recent paper [62].

In [28], it has already been argued that space-time white noise is not compatible with the finiteness of the physical energies encountered in thin-film flow. Therefore, we consider  $Q$ -Wiener processes. To guarantee conservation of mass, we work on rectangular physical domains  $\mathcal{O} := \mathcal{O}^x \times \mathcal{O}^y := (0, L_x) \times (0, L_y)$ , and we prescribe periodic boundary conditions. We may consider an ON-basis  $(\mathbf{g}_{kl})_{k,l \in \mathbb{N}}$  where the  $\mathbf{g}_{kl}$  are given as the product  $\mathbf{g}_{kl}(x, y) := g_k^x(x)g_l^y(y)$  of appropriately scaled eigen-functions of the one-dimensional Laplace operator on  $\mathcal{O}^x$  and  $\mathcal{O}^y$ , respectively (cf. Remark 3.2).

We consider driving noise  $\mathbf{W}$  given by

$$\begin{pmatrix} \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^x \mathfrak{g}_{kl} \beta_{kl}^x \\ \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^y \mathfrak{g}_{kl} \beta_{kl}^y \end{pmatrix} \quad (1.2)$$

where

- the  $\beta_{kl}^\alpha$ ,  $\alpha \in \{x, y\}$ ,  $k, l \in \mathbb{Z}$  constitute a family of i.i.d. Brownian motions,
- the  $\lambda_{kl}^\alpha$ ,  $\alpha \in \{x, y\}$ ,  $k, l \in \mathbb{Z}$  are a family of nonnegative real numbers converging sufficiently fast to zero – see Hypothesis **(B3)** for more details.

Therefore, we are interested in global existence of a.s. nonnegative martingale solutions to the stochastic thin-film equation

$$du = -\operatorname{div} \{m(u) \nabla (\Delta u - F'(u))\} dt + \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \partial_\alpha (\sqrt{m(u)} \lambda_{kl}^\alpha \mathfrak{g}_{kl}) d\beta_{kl}^\alpha \quad (1.3)$$

on  $\mathcal{O} \times [0, \infty)$  subject to periodic boundary conditions.

As the analysis of stochastic thin-film equations is influenced by the deterministic theory, we give a brief account on the literature, following here the exposition in [16].

A theory of existence of weak solutions for the deterministic thin-film equation in space dimension  $d = 1$  has been developed in [5, 3, 6] and [60, 8, 58] for zero and nonzero contact angles at the intersection of the liquid-gas and liquid-solid interfaces, respectively, while the multi-dimensional version with  $F \equiv 0$  in  $\mathcal{O} \times (0, T]$  and zero contact angles has been the subject of [14, 45]. For these solutions, a number of quantitative results has been obtained – including optimal estimates on spreading rates of free boundaries, i.e. the triple lines separating liquid, gas, and solid, see [50, 7, 42, 25], optimal conditions on the occurrence of waiting time phenomena [15], as well as scaling laws for the size of waiting times [33, 26]. For the deterministic case with  $F \neq 0$ , we refer to [44] for an existence result based on numerical analysis.

A corresponding theory of classical solutions, giving the existence and uniqueness for initial data close to generic solutions or short times, has been developed in [35, 34, 32, 37, 38, 36] for zero contact angles and in [54, 56, 57, 55, 24] for nonzero contact angles in one space dimension, while the higher-dimensional version has been the subject of [52, 63, 39] and [18] for zero and nonzero contact angles, respectively.

The outline of our paper is as follows. Conceptually, our existence result is based on stochastic counterparts of integral estimates known from the deterministic setting which we combine with Jakubowski/Skorokhod-type methods to construct martingale solutions – see [11, 49] for the basic ideas and [48, 9] for applications to other problems. More precisely, we will control the energy

$$\mathcal{E}(u) := \int_{\mathcal{O}} \frac{1}{2} |\nabla u|^2 dx dy + \int_{\mathcal{O}} F(u) dx dy \quad (1.4)$$

and the so called mathematical entropy

$$\mathcal{S}(u) := \int_{\mathcal{O}} G(u) dx dy \quad (1.5)$$

where

$$G(u) := \int_1^u \int_1^s \frac{1}{m(r)} dr ds \quad (1.6)$$

is a second primitive of the reciprocal mobility.

To derive those integral estimates, we mimic the 1d strategy used in [28], i.e. we perform discretization in space and apply Itô's formula to the resulting system of SDEs. Extending this 1D approach (in particular the treatment of the additional terms arising from Itô's formula) to the two-dimensional setting requires the use of tensor product finite elements. We adapt the stopping-time-approach of [28] to comply with the singularities in the effective interface potential  $F(\cdot)$ . In contrast to the spatially one-dimensional setting of [28], boundedness of the physical energy  $\mathcal{E}(u)$  does no longer imply strict positivity in our case. Therefore, in the discrete setting,  $\mathcal{E}(u)$  is augmented by the square of the  $L^2$ -norm of the discrete Laplacian of discrete solutions  $u_h$ , weighted by a factor which vanishes in the limit  $h \rightarrow 0$ . For the details, see (3.1) and (2.7).

Section 2 is devoted to notation and the large number of technical preliminaries which come along with our discretization and the energy regularization mentioned before. Moreover, the assumptions on initial data, growth behavior of the effective interface potential  $F$  and on the driving noise  $\mathbf{W}$  are specified in Section 2, too.

Section 3 is devoted to the discussion of the semi-discrete scheme which is formulated in such a way that it may be used for practical numerical simulations of stochastic thin-film equations as well. Section 3 contains also the main existence result together with the applied solution concept. Moreover, we present a lemma which permits the control of the oscillation of discrete solutions on single finite elements.

In Section 4, we present the core result of the analysis in this paper – a discrete combined energy-entropy estimate. It is based on a combination of Itô's formula with inverse estimates for finite-element functions, and with error estimates for interpolation operators which are collected in a synopsis in Appendix A. Moreover, Section 4 contains results on compactness in time of discrete solutions which – in combination with the aforementioned energy-entropy estimate – are the key to apply the Skorokhod-Jakubowski-method (cf. [11, 49, 51]) to pass to the limit  $h \rightarrow 0$  which is the topic of Section 5.

It is worth mentioning that the passage to the limit in the deterministic terms poses new intricacies due to the lack of strict positivity results. We base our arguments on appropriate exhaustion arguments combined with generalizations of Egorov's theorem for Bochner-integrable functions.

Notation: Throughout the paper, we use the standard notation for Sobolev spaces, i.e. for a spatial domain  $\mathcal{O} \subset \mathbb{R}^2$ , we denote the space of  $k$ -times weakly differentiable functions with weak derivatives in  $L^p(\mathcal{O})$  by  $W^{k,p}(\mathcal{O})$ . For  $p = 2$ , we denote the Hilbert spaces  $W^{k,2}(\mathcal{O})$  by  $H^k(\mathcal{O})$ . The corresponding subspaces of  $\mathcal{O}$ -periodic functions will be denoted by the subscript 'per'. The subspace of  $\mathcal{O}$ -periodic  $H^1(\mathcal{O})$ -functions with mean-value zero will be denoted by  $H_*^1(\mathcal{O})$ . Furthermore, we denote the dual space of  $H_{\text{per}}^1(\mathcal{O})$  by  $(H_{\text{per}}^1(\mathcal{O}))'$ . The space of continuous  $\mathcal{O}$ -periodic functions is denoted by  $C_{\text{per}}(\overline{\mathcal{O}})$  and  $C_{\text{per}}^\gamma(\overline{\mathcal{O}})$  is the space of  $\mathcal{O}$ -periodic, Hölder continuous functions with Hölder exponent  $\gamma$ . For a time interval  $I$  and a Banach space  $X$ , the space of  $L^p$ -integrable functions with values in  $X$  is denoted by  $L^p(I; X)$ . Similarly, we denote the space of  $k$ -times weakly differentiable functions from  $I$  to  $X$  with weak derivatives in  $L^p(I; X)$  by  $W^{k,p}(I; X)$  and the Hölder continuous functions from  $I$  to  $X$  with Hölder exponent  $\gamma$  by  $C^\gamma(I; X)$ .

We shall also use some standard notation from stochastic analysis: The notation  $a \wedge b$  stands for the minimum of  $a$  and  $b$ , and  $L_2(X, Y)$  denotes the set of Hilbert-Schmidt operators from  $X$  to  $Y$ . For a stopping time  $T$ , we write  $\chi_T$  to denote the ( $\omega$ -dependent) characteristic function of the time interval  $[0, T]$ .

Further notation related to the semi-discrete scheme is introduced in Section 2.

## 2. NOTATION, TECHNICAL PRELIMINARIES, AND BASIC ASSUMPTIONS ON THE DATA

We consider the torus  $\mathcal{O} := \mathcal{O}^x \times \mathcal{O}^y := (0, L_x) \times (0, L_y)$ . We introduce partitions  $\mathcal{T}_h^x$  and  $\mathcal{T}_h^y$  of  $\mathcal{O}^x$  and  $\mathcal{O}^y$  satisfying the following assumption:

- (S)  $\{\mathcal{T}_h^x\}_{h>0}$  and  $\{\mathcal{T}_h^y\}_{h>0}$  are families of equidistant partitions of  $\mathcal{O}^x$  and  $\mathcal{O}^y$ , respectively, into disjoint, open intervals such that

$$\overline{\mathcal{O}^x} \equiv \bigcup_{K^x \in \mathcal{T}_h^x} \overline{K^x} \quad \text{and} \quad \overline{\mathcal{O}^y} \equiv \bigcup_{K^y \in \mathcal{T}_h^y} \overline{K^y}.$$

In particular, there exist positive constants  $\hat{c}_1, \hat{C}_2$  such that

$$\hat{c}_1 h \leq h_x, h_y \leq \hat{C}_2 h$$

with  $h_x := \text{diam } K^x$  ( $K^x \in \mathcal{T}_h^x$ ),  $h_y := \text{diam } K^y$  ( $K^y \in \mathcal{T}_h^y$ ), and  $h \in (0, 1)$ .

Combining  $\{\mathcal{T}_h^x\}_h$  and  $\{\mathcal{T}_h^y\}_h$ , we obtain a family of partitions  $\{\mathcal{Q}_h\}_h$  of  $\mathcal{O}$  which is defined via

$$\mathcal{Q}_h = \{Q = K^x \times K^y : K^x \in \mathcal{T}_h^x \text{ and } K^y \in \mathcal{T}_h^y\}. \quad (2.1)$$

Based on these partitions, we introduce the following spaces of continuous, piecewise linear finite element functions.

$$U_h^x := \{v \in C_{\text{per}}(\overline{\mathcal{O}^x}) : v|_{K^x} \in \mathcal{P}_1(K^x) \quad \forall K^x \in \mathcal{T}_h^x\}, \quad (2.2a)$$

$$U_h^y := \{v \in C_{\text{per}}(\overline{\mathcal{O}^y}) : v|_{K^y} \in \mathcal{P}_1(K^y) \quad \forall K^y \in \mathcal{T}_h^y\}, \quad (2.2b)$$

$$U_h := U_h^x \otimes U_h^y. \quad (2.2c)$$

Imposing periodic boundary conditions, we denote the vertices of  $\mathcal{T}_h^x$  by  $\{x_i\}_{i=1, \dots, \dim U_h^x+1} = \{(i-1)h_x\}_{i=1, \dots, \dim U_h^x+1}$  and identify  $x_{\dim U_h^x+1} = L_x$  with  $x_1 = 0$ . Furthermore, we denote the dual basis to these vertices by  $\{\mathbf{e}_i^x\}_{i=1, \dots, \dim U_h^x}$ . Similarly, we denote the vertices of  $\mathcal{T}_h^y$  by  $\{y_j\}_{j=1, \dots, \dim U_h^y+1}$ , identify  $y_{\dim U_h^y+1} = L_y$  with  $y_1 = 0$ , and consider their dual basis  $\{\mathbf{e}_j^y\}_{j=1, \dots, \dim U_h^y}$ . Throughout this work, we will also identify  $x_0$  with  $x_{\dim U_h^x}$  and  $y_0$  with  $y_{\dim U_h^y}$ . In the same spirit, we shall identify  $\mathbf{e}_{\dim U_h^x+1}^x$  with  $\mathbf{e}_1^x$ ,  $\mathbf{e}_0^x$  with  $\mathbf{e}_{\dim U_h^x}^x$ ,  $\mathbf{e}_{\dim U_h^y+1}^y$  with  $\mathbf{e}_1^y$ , and  $\mathbf{e}_0^y$  with  $\mathbf{e}_{\dim U_h^y}^y$ . For the spaces introduced in (2.2), we define the interpolation operators

$$\mathcal{I}_h^x : C_{\text{per}}(\overline{\mathcal{O}^x}) \rightarrow U_h^x \quad a \mapsto \sum_{i=1}^{\dim U_h^x} a(x_i) \mathbf{e}_i^x, \quad (2.3)$$

$$\mathcal{I}_h^y : C_{\text{per}}(\overline{\mathcal{O}^y}) \rightarrow U_h^y \quad a \mapsto \sum_{j=1}^{\dim U_h^y} a(y_j) \mathbf{e}_j^y, \quad (2.4)$$

$$\mathcal{I}_h^{xy} : C_{\text{per}}(\overline{\mathcal{O}}) \rightarrow U_h \quad a \mapsto \mathcal{I}_h^x \{\mathcal{I}_h^y \{a\}\} = \mathcal{I}_h^y \{\mathcal{I}_h^x \{a\}\}. \quad (2.5)$$

For future reference, we state the following norm equivalence for  $p \in [1, \infty)$  and  $u_h \in U_h$ :

$$c \left( \int_{\mathcal{O}} (u_h)^p \, dx \, dy \right)^{1/p} \leq \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{(u_h)^p\} \, dx \, dy \right)^{1/p} \leq C \left( \int_{\mathcal{O}} (u_h)^p \, dx \, dy \right)^{1/p} \quad (2.6)$$

with  $c, C > 0$  independent of  $h$ . Similar (lower dimensional) results also hold true for  $\mathcal{I}_h^x$  on  $U_h^x$  and  $\mathcal{I}_h^y$  on  $U_h^y$ . These nodal interpolation operators satisfy error estimates similar to the ones established in [59] for simplicial elements. For the reader's convenience, we collect these estimates in Lemma A.1 in the appendix.

With these interpolation operators, we define the discrete Laplacian as follows:

$$\begin{aligned} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{-\Delta_h u_h \psi_h\} dx dy &:= \int_{\mathcal{O}} \mathcal{I}_h^y \{\partial_x u_h \partial_x \psi_h\} dx dy + \int_{\mathcal{O}} \mathcal{I}_h^x \{\partial_y u_h \partial_y \psi_h\} dx dy \\ &=: - \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{\Delta_h^x u_h \psi_h\} dx dy - \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{\Delta_h^y u_h \psi_h\} dx dy \end{aligned} \quad (2.7)$$

for all  $\psi \in U_h$ .

We denote the forward and backward difference quotients w.r.t. the spatial coordinates  $x$  and  $y$  by  $\partial_x^{+h_x}$ ,  $\partial_x^{-h_x}$ ,  $\partial_y^{+h_y}$ , and  $\partial_y^{-h_y}$ , i.e.

$$\partial_x^{+h_x} f(x, y) := (f(x + h_x, y) - f(x, y))/h_x, \quad (2.8a)$$

$$\partial_x^{-h_x} f(x, y) := (f(x, y) - f(x - h_x, y))/h_x, \quad (2.8b)$$

$$\partial_y^{+h_y} f(x, y) := (f(x, y + h_y) - f(x, y))/h_y, \quad (2.8c)$$

$$\partial_y^{-h_y} f(x, y) := (f(x, y) - f(x, y - h_y))/h_y \quad (2.8d)$$

(with  $f$  extended outside of  $\mathcal{O}$  by periodicity). Assuming equidistant partitions w.r.t.  $x$  and  $y$ , the identities

$$\Delta_h^x v_h = \partial_x^{+h_x} (\partial_x^{-h_x} v_h) = \partial_x^{-h_x} (\partial_x^{+h_x} v_h), \quad (2.9a)$$

$$\Delta_h^y v_h = \partial_y^{+h_y} (\partial_y^{-h_y} v_h) = \partial_y^{-h_y} (\partial_y^{+h_y} v_h) \quad (2.9b)$$

hold true for  $v_h \in U_h$ .

In addition, we introduce similar local interpolation operators as follows. We consider the spaces

$$C_{\text{per}, \mathcal{T}_h^x} := \{v \in L_{\text{per}}^\infty(\mathcal{O}^x) : v|_{K^x} \in C(K^x) \quad \forall K^x \in \mathcal{T}_h^x\}, \quad (2.10)$$

$$C_{\text{per}, \mathcal{T}_h^y} := \{v \in L_{\text{per}}^\infty(\mathcal{O}^y) : v|_{K^y} \in C(K^y) \quad \forall K^y \in \mathcal{T}_h^y\} \quad (2.11)$$

of bounded, piecewise continuous, periodic functions. As we can extend a continuous function on an open interval to a continuous function on the closure of this interval, we may apply  $\mathcal{I}_h^x$  and  $\mathcal{I}_h^y$  locally on each element to obtain

$$\mathcal{I}_{h, \text{loc}}^x : C_{\text{per}, \mathcal{T}_h^x} \rightarrow \{v \in C_{\text{per}, \mathcal{T}_h^x} : v|_{K^x} \in \mathcal{P}_1(K^x) \quad \forall K^x \in \mathcal{T}_h^x\}, \quad (2.12a)$$

$$\mathcal{I}_{h, \text{loc}}^y : C_{\text{per}, \mathcal{T}_h^y} \rightarrow \{v \in C_{\text{per}, \mathcal{T}_h^y} : v|_{K^y} \in \mathcal{P}_1(K^y) \quad \forall K^y \in \mathcal{T}_h^y\}. \quad (2.12b)$$

Obviously, these local interpolation operators satisfy the identities

$$\mathcal{I}_{h, \text{loc}}^x \{\partial_x a_h^x v\} = \partial_x a_h^x \mathcal{I}_{h, \text{loc}}^x \{v\}, \quad \text{and} \quad \mathcal{I}_{h, \text{loc}}^y \{\partial_y a_h^y \hat{v}\} = \partial_y a_h^y \mathcal{I}_{h, \text{loc}}^y \{\hat{v}\}, \quad (2.13a)$$

$$\mathcal{I}_h^x \{\tilde{v}\} = \mathcal{I}_{h, \text{loc}}^x \{\tilde{v}\}, \quad \text{and} \quad \mathcal{I}_h^y \{\bar{v}\} = \mathcal{I}_{h, \text{loc}}^y \{\bar{v}\} \quad (2.13b)$$

for  $v \in C_{\text{per}, \mathcal{T}_h^x}$ ,  $\hat{v} \in C_{\text{per}, \mathcal{T}_h^y}$ ,  $\tilde{v} \in C_{\text{per}}(\overline{\mathcal{O}^x})$ , and  $\bar{v} \in C_{\text{per}}(\overline{\mathcal{O}^y})$ . Here, the first identity follows directly from the definition of  $\mathcal{I}_{h, \text{loc}}^x$ , as  $\partial_x a_h^x$  is constant w.r.t.  $x$  on every element.

In order to allow for a discrete version of the chain rule, we introduce for a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $u \in C_{\text{per}}(\overline{\mathcal{O}})$  for every  $y \in \mathcal{O}^y$  and  $x \in (ih, (i+1)h) =: K^x \in \mathcal{T}_h^x$

the function

$$[f(u)]_x(x, y) := \int_{u(ih, y)}^{u((i+1)h, y)} f(s) ds. \quad (2.14)$$

Similarly, we define for  $x \in \mathcal{O}^x$  and  $y \in (jh, (j+1)h) =: K^y \in \mathcal{T}_h^y$

$$[f(u)]_y(x, y) := \int_{u(x, jh)}^{u(x, (j+1)h)} f(s) ds. \quad (2.15)$$

Obviously, these definitions provide for  $u_h \in U_h$

$$\partial_x \mathcal{I}_h^{xy} \{f(u_h)\} = \mathcal{I}_h^y \{[f'(u_h)]_x \partial_x u_h\}, \quad \partial_y \mathcal{I}_h^{xy} \{f(u_h)\} = \mathcal{I}_h^x \{[f'(u_h)]_y \partial_y u_h\}. \quad (2.16)$$

On the periodic domain  $\mathcal{O}$ , we define the Ritz projection operator  $\mathcal{R} : H_{\text{per}}^1(\mathcal{O}) \rightarrow U_h$  via

$$\int_{\mathcal{O}} \nabla \mathcal{R}\{u\} \cdot \nabla v_h dx dy = \int_{\mathcal{O}} \nabla u \cdot \nabla v dx dy \quad \text{for all } v_h \in U_h \quad (2.17)$$

with the additional constraint  $\int_{\mathcal{O}} \mathcal{R}\{u\} dx dy = \int_{\mathcal{O}} u dx dy$ .

Let us specify our assumptions on initial data, effective interface potential, and the noise.

- (I) Let  $\Lambda$  be a probability measure on  $H_{\text{per}}^2(\mathcal{O})$  equipped with the Borel  $\sigma$ -algebra which is supported on the subset of strictly positive functions such that there is a positive constant  $C$  with the property

$$\text{esssup}_{v \in \text{supp } \Lambda} \left( \mathcal{E}_h(\mathcal{I}_h^{xy}\{v\}) + \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{v\} dx dy + \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy}\{v\} dx dy \right)^{-1} \right) \leq C \quad (2.18)$$

for any  $h > 0$  with  $\mathcal{E}_h$  being a discrete version of the energy (1.4), which we will define in (3.1).

- (P) The effective interface potential  $F$  has continuous second-order derivatives on  $\mathbb{R}^+$  and satisfies for some  $p > 2$  and  $u > 0$  the following estimates with appropriate positive constants:

$$\begin{aligned} F(u) &\geq c_1 u^{-p}, \\ |F'(u)| &\leq \hat{C} u^{-p-1} + \hat{C}, \\ \tilde{c}_1 u^{-p-2} - \tilde{c}_2 &\leq F''(u) \leq \tilde{C} u^{-p-2} + \tilde{C}. \end{aligned}$$

For nonpositive  $u$ , we define  $F(u) := +\infty$ .

- (B) Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a stochastic basis with a complete, right-continuous filtration such that

- (B1)  $\mathbf{W}$  is a  $Q$ -Wiener process on  $\Omega$  adapted to  $(\mathcal{F}_t)_{t \geq 0}$  which admits a decomposition of the form  $\mathbf{W} = \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^\alpha \mathbf{g}_{kl} \mathbf{b}_\alpha \beta_{kl}^\alpha$  for independent sequences of i.i.d. Brownian motions  $\beta_{kl}^\alpha$  ( $\alpha \in \{x, y\}$ ) and a sequence of sufficiently smooth basis functions  $\mathbf{g}_{kl}$ . Here,  $\mathbf{b}_x$  and  $\mathbf{b}_y$  denote the standard Cartesian basis vectors in  $\mathbb{R}^2$ . Furthermore, we will denote its components by  $W_\alpha := \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^\alpha \mathbf{g}_{kl} \beta_{kl}^\alpha$  ( $\alpha \in \{x, y\}$ ). The corresponding components of  $Q$  will be denoted by  $Q_x$  and  $Q_y$ .

- (B2) there exists a  $\mathcal{F}_0$ -measurable random variable  $u^0$  such that  $\Lambda = \mathbb{P} \circ (u^0)^{-1}$ .

**(B3)** the noise  $\mathbf{W}$  is colored in the sense that  $\sum_{k,l=1}^{\infty} \left( \lambda_{kl}^x{}^2 + \lambda_{kl}^y{}^2 \right) \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \leq C$  for a positive constant  $C$ .

**Remark 2.1.** (1) Under natural assumptions on the decay parameters  $\lambda_{kl}^\alpha$  and the basis functions  $\mathbf{g}_{kl}$ ,  $k, l \in \mathbb{Z}$ , Hypothesis **(P)** covers in fact also the case that the stochastic integral is to be understood in the sense of Stratonovich. Assuming

- the basis functions to be given by  $\mathbf{g}_{kl}(x, y) = g_k^x(x)g_l^y(y)$  with  $g_k^x(\cdot)$  and  $g_l^y(\cdot)$  as in (3.4),
- decay parameters  $\lambda_{kl}^\alpha$ ,  $k, l \in \mathbb{Z}$ ,  $\alpha \in \{x, y\}$  to satisfy

$$\lambda_{kl}^x = \lambda_{kl}^y, \quad \lambda_{(-k)l}^\alpha = \lambda_{kl}^\alpha, \quad \lambda_{k(-l)}^\alpha = \lambda_{kl}^\alpha \quad (2.19)$$

for all  $k, l \in \mathbb{Z}$  and  $\alpha \in \{x, y\}$ ,

the Itô correction of the Stratonovich term

$$\sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \partial_\alpha (u \lambda_{kl}^\alpha \mathbf{g}_{kl}) \circ d\beta_{kl}^\alpha$$

becomes

$$C_{Strat} \Delta u \, dt + \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \partial_\alpha (u \lambda_{kl}^\alpha \mathbf{g}_{kl}) \, d\beta_{kl}^\alpha. \quad (2.20)$$

Here, the positive constant  $C_{Strat}$  is given by

$$C_{Strat} := \frac{1}{L_x L_y} \left( \lambda_{00}^2 + 4 \sum_{k, l \in \mathbb{Z} \setminus \{0\}} \lambda_{kl}^2 + 2 \sum_{k \in \mathbb{Z} \setminus \{0\}} (\lambda_{k0}^2 + \lambda_{0k}^2) \right), \quad (2.21)$$

where we omitted the superscript  $\alpha$  as the decay parameters were chosen to be independent of  $\alpha$ . This allows to write the stochastic thin-film equation with Stratonovich noise in the form

$$du = -\operatorname{div} (u^2 \nabla (\Delta u - F'_{Strat}(u))) \, dt + \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \partial_\alpha (u \lambda_{kl} \mathbf{g}_{kl}) \, d\beta_{kl}^\alpha \quad (2.22)$$

with the energy  $F_{Strat}$  given by

$$F_{Strat}(u) := \begin{cases} F(u) + C_{Strat} (u - \log u) + \text{const.} & \text{if } u > 0 \\ +\infty & \text{if } u \leq 0 \end{cases} \quad (2.23)$$

where the constant can be chosen in such a way that  $F_{Strat}$  satisfies **(P)** if **(P)** is satisfied by  $F$  itself. Hence, the analysis presented in this paper applies to the Stratonovich interpretation, too.

(2) The approximation of initial data is based on the nodal interpolation operator to cope with the requirement of strictly positive discrete initial data. Therefore, we need the space of initial data to be continuously embedded in  $C(\overline{\mathcal{O}})$ . The specification in **(I)** that initial data should have  $H^2$ -regularity is presumably not the optimal one. It is, however, consistent with our regularization procedure – see (3.6b) – of augmenting the pressure  $p_h$  by a discrete Bi-Laplacian. As the energy estimate formulated for (1.3) (see (3.9)) does not require more than  $H^1$ -regularity for initial data, it should have been possible to focus on  $H^1$ -initial data and to apply nodal interpolation operators to appropriate  $H^2$ -regularizations, e.g. by convolution. For the ease of presentation, we prefer to avoid those technicalities.

### 3. THE SEMI-DISCRETE SCHEME

In order to control the oscillation of the discrete solution  $u_h$  on each element, we regularize the energy under consideration. Introducing a regularization parameter  $2 > \varepsilon > 0$ , we define the regularized discrete energy and the discrete entropy as

$$\begin{aligned} \mathcal{E}_h(u_h) &:= \frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} + \mathcal{I}_h^x \{ |\partial_y u_h|^2 \} dx dy + \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(u_h) \} dx dy \\ &\quad + \frac{1}{2} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy, \end{aligned} \quad (3.1)$$

$$\mathcal{S}_h(u_h) := \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ G(u_h) \} dx dy \quad \text{with} \quad G(s) := \int_1^s \int_1^r \frac{1}{m(\tau)} d\tau dr. \quad (3.2)$$

As it will be shown in Lemma 3.6, we assume that

(**R**) the regularization parameter  $\varepsilon$  is small enough such that there exists a constant  $\rho > 0$  such that

$$1 > \frac{2}{p} + \frac{\varepsilon}{2} + \frac{\rho}{2p}, \quad (3.3)$$

where  $p$  is the exponent associated with the growth of  $F$  (cf. Assumption (**P**)).

**Remark 3.1.** For every exponent  $p > 2$  in the effective interface potential, positive parameters  $\varepsilon$  and  $\rho$  exist, such that (**R**) holds true.

Given a positive time  $T_{\max}$ , we introduce a threshold energy  $\mathcal{E}_{\max,h} := \hat{C} h^{-\rho/(2+p)}$  for given  $\hat{C} > 0$  and  $0 < \rho \ll 1$  satisfying (**R**). Similarly as in [28], we consider associated stopping times  $T_h := T_{\max} \wedge \inf \{ t \geq 0 : \mathcal{E}(u_h) \geq \mathcal{E}_{\max,h} \}$ . We approximate the infinite dimensional Wiener process by a finite dimensional noise term. In particular, we introduce the sets  $I_h^x \subset \mathbb{Z}$  and  $I_h^y \subset \mathbb{Z}$  satisfying

(**B3\***)  $\sum_{k \in I_h^x} \sum_{l \in I_h^y} (\lambda_{kl}^x{}^2 + \lambda_{kl}^y{}^2) h^\varepsilon \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \leq C$  for  $h \searrow 0$ ,

(**B4**)  $I_h^x \subseteq I_{\hat{h}}^x$ ,  $I_h^y \subseteq I_{\hat{h}}^y$  for  $h \geq \hat{h}$  and  $\bigcup_{h>0} I_h^x = \mathbb{Z}$  and  $\bigcup_{h>0} I_h^y = \mathbb{Z}$ .

**Remark 3.2.** Often the basis functions  $\mathbf{g}_{kl}$  are assumed to be eigenfunctions of the negative Laplacian on  $\mathcal{O}$  under periodic boundary conditions. In particular, the functions  $\mathbf{g}_{kl}$  are assumed to be the product of eigenfunctions  $g_k^x$  and  $g_l^y$  of the one-dimensional Laplacian on  $\mathcal{O}^x$  and  $\mathcal{O}^y$ , respectively, i.e.

$$g_k^x(x) := \sqrt{\frac{2}{L_x}} \begin{cases} \cos\left(\frac{2\pi kx}{L_x}\right) & \text{for } k \geq 1, \\ \frac{1}{\sqrt{2}} & \text{for } k = 0, \\ \sin\left(\frac{2\pi kx}{L_x}\right) & \text{for } k \leq -1, \end{cases} \quad g_l^y(y) := \sqrt{\frac{2}{L_y}} \begin{cases} \cos\left(\frac{2\pi ly}{L_y}\right) & \text{for } l \geq 1, \\ \frac{1}{\sqrt{2}} & \text{for } l = 0, \\ \sin\left(\frac{2\pi ly}{L_y}\right) & \text{for } l \leq -1. \end{cases} \quad (3.4)$$

In this case, we have  $\|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})} \sim (k^2 + l^2)$  and  $\|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})} \leq C(k^3 + l^3)$ . Therefore, one may choose  $I_h^x = I_h^y = \{z \in \mathbb{Z} : |z| \leq \hat{C} h^{-\varepsilon/2}\}$  for given  $\hat{C} > 0$  to satisfy Assumptions (**B3\***) and (**B4**), i.e. the additional restrictions on the noise term imposed in (**B3\***) vanish when passing to the limit  $h \searrow 0$ .

With the perspective to simplify the implementation in a practical numerical scheme, we approximate the basis functions  $\mathbf{g}_{kl}$  by  $\tilde{\mathbf{g}}_{h,kl} := \mathcal{I}_h^{xy} \{\mathbf{g}_{kl}\}$ .

In this work, we consider solutions

$$u_h \in L^2(\Omega; C([0, T_{\max}]; U_h)), \quad (3.5a)$$

$$p_h \in L^2(\Omega; L^\infty(0, T_{\max}; U_h)) \quad (3.5b)$$

to the following regularized, semi-discrete version of (1.3):

$$\begin{aligned} & \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{u_h(T)\psi_h\} \, dx \, dy - \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{u_h(0)\psi_h\} \, dx \, dy \\ & \quad + \int_0^{T \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x \psi_h \right\} \, dx \, dy \, dt \\ & \quad + \int_0^{T \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y \psi_h \right\} \, dx \, dy \, dt \\ & = \sum_{k \in I_h^x} \sum_{k \in I_h^y} \lambda_{kl}^x \int_0^{T \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \psi_h \} \right\} \, dx \, dy \, d\beta_{kl}^x \\ & \quad + \sum_{k \in I_h^x} \sum_{k \in I_h^y} \lambda_{kl}^y \int_0^{T \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathbf{g}}_{h,kl}) \psi_h \} \right\} \, dx \, dy \, d\beta_{kl}^y \end{aligned} \quad (3.6a)$$

$$\begin{aligned} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{p_h \psi_h\} \, dx \, dy &= \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \partial_x \psi_h \} \, dx \, dy + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \partial_y u_h \partial_y \psi_h \} \, dx \, dy \\ & \quad + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{F'(u_h) \psi_h\} \, dx \, dy + \chi_{T_h} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h u_h \Delta_h \psi_h \} \, dx \, dy. \end{aligned} \quad (3.6b)$$

**Remark 3.3.** Note that for discrete solutions of (3.6a) the mass of discrete solutions, i.e.  $\int_{\mathcal{O}} u_h(t) \, dx \, dy = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{u_h(t)\} \, dx \, dy$ , is constant in time. Of course, it is natural to choose  $\psi_h \equiv 1$  as the test function in (3.6a). Obviously, the contribution by the elliptic terms vanishes. So let us briefly prove that also the stochastic terms become zero. Using (2.13a) and both  $u_h$  and  $\tilde{\mathbf{g}}_{h,kl}$  to be contained in  $U_h$ , we find

$$\begin{aligned} & \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \} \right\} \, dx \, dy \\ & = \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x u_h \mathcal{I}_{h,\text{loc}}^x \{ \tilde{\mathbf{g}}_{h,kl} \} + \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_{h,\text{loc}}^x \{ u_h \} \right\} \, dx \, dy \\ & = \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \} \, dx \, dy = 0 \end{aligned}$$

due to integration by parts.

**Definition 3.4.** Let  $\Lambda$  be a probability measure on  $H_{\text{per}}^2(\mathcal{O})$  satisfying (I). A triple  $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{u}, \tilde{\mathbf{W}})$  is called a weak martingale solution to the stochastic thin-film equation (1.3) with initial data  $\Lambda$  on the time interval  $[0, T_{\max}]$  provided

- (1)  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$  is a stochastic basis with a complete, right-continuous filtration,
- (2)  $\tilde{\mathbf{W}}$  satisfies Assumption (B) with respect to  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ ,

(3) the solution  $\tilde{u}$  is element of

$$\begin{aligned} L^q(\tilde{\Omega}; L^\infty(0, T_{max}; H_{\text{per}}^1(\mathcal{O}))) \cap L^2(\tilde{\Omega}; L^2(0, T_{max}; H_{\text{per}}^2(\mathcal{O}))) \\ \cap L^\sigma(\tilde{\Omega}; C^{1/4}([0, T_{max}]; (H_{\text{per}}^1(\mathcal{O}))')) \end{aligned} \quad (3.7)$$

for all  $q < \infty$  and  $\sigma < 8/5$  such that  $\sqrt{m(\tilde{u})} \nabla(\Delta \tilde{u} - F'(\tilde{u})) \in L^2([\tilde{u} > 0])$ ,

(4) there exists an  $\tilde{F}_0$ -measurable  $H_{\text{per}}^2(\mathcal{O}; \mathbb{R}^+)$ -valued random variable  $\tilde{u}^0$  such that  $\Lambda = \tilde{\mathbb{P}} \circ (\tilde{u}^0)^{-1}$ , and the equation

$$\begin{aligned} \int_{\mathcal{O}} \tilde{u}(t) \phi \, dx \, dy = \int_{\mathcal{O}} \tilde{u}^0 \phi \, dx \, dy + \int \int_{[\tilde{u} > 0]} m(\tilde{u}) \nabla(\Delta \tilde{u} - F'(\tilde{u})) \cdot \nabla \phi \, dx \, dy \, ds \\ - \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^\alpha \int_0^t \int_{\mathcal{O}} \sqrt{m(\tilde{u})} \mathbf{g}_{kl} \partial_\alpha \phi \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \end{aligned} \quad (3.8)$$

holds true  $\tilde{\mathbb{P}}$ -almost surely for all  $t \in [0, T_{max}]$  and all  $\phi \in W_{\text{per}}^{1, q^*}(\mathcal{O})$  with  $q^* > 2$ .

The aim of this work is to establish the existence of weak martingale solutions starting from semi-discrete solutions to (3.6). In particular, we shall prove the following theorem.

**Theorem 3.5.** *Let Assumptions (S), (I), (P), (B), (R), (B3\*), and (B4) be satisfied and let  $T_{max} > 0$  be given. Furthermore, let  $(u_h, p_h)_{h \searrow 0}$  be a sequence of solutions to the regularized Faedo-Galerkin scheme (3.6) for the stochastic thin-film equation (1.3) with  $\mathcal{E}_{max, h} = \hat{C} h^{-\rho/(2+p)}$  for some given  $\hat{C} > 0$ .*

*Then there exist a stochastic basis  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$  as well as processes  $\tilde{u}_h, \tilde{J}_h^x, \tilde{J}_h^y$ , and  $\tilde{u}$  such that the following holds: The processes  $\tilde{u}_h, \tilde{J}_h^x$ , and  $\tilde{J}_h^y$  have the same law as the processes  $u_h, J_h^x := \mathcal{I}_h^y \left\{ \sqrt{[G''(u_h)]_x^{-1} \partial_x p_h} \right\}$ , and  $J_h^y := \mathcal{I}_h^x \left\{ \sqrt{[G''(u_h)]_y^{-1} \partial_y p_h} \right\}$  and for a subsequence we  $\tilde{\mathbb{P}}$ -almost surely have the convergences  $\tilde{u}_h \rightarrow \tilde{u}$  strongly in  $C([0, T_{max}]; L^q(\mathcal{O})) \cap L^2(0, T_{max}; W_{\text{per}}^{1, q}(\mathcal{O}))$  ( $q < \infty$ ),  $\tilde{J}_h^x \rightharpoonup \tilde{J}^x$  weakly in  $L^2(0, T_{max}; L^2(\mathcal{O}))$ , which can be identified with  $-\tilde{u} \partial_x(\Delta \tilde{u} - F'(\tilde{u}))$  on  $[\tilde{u} > 0]$ , and  $\tilde{J}_h^y \rightharpoonup \tilde{J}^y$  weakly in  $L^2(0, T_{max}; L^2(\mathcal{O}))$ , which can be identified with  $-\tilde{u} \partial_y(\Delta \tilde{u} - F'(\tilde{u}))$  on  $[\tilde{u} > 0]$ . Furthermore,  $\tilde{u}$  is a weak martingale solution to the stochastic thin-film equation in the sense of Definition 3.4 satisfying the additional bound*

$$\tilde{\mathbb{E}} \left[ \sup_{t \in [0, T_{max}]} (\mathcal{E}(\tilde{u}))^{\bar{p}} \right] + \tilde{\mathbb{E}} \left[ \int \int_{[\tilde{u} > 0]} m(\tilde{u}) |\nabla(\Delta \tilde{u} - F'(\tilde{u}))|^2 \, dx \, dy \, dt \right] \leq C(u^0, \bar{p}, T_{max}) \quad (3.9)$$

with  $\bar{p} < \infty$ . In particular,  $\tilde{\mathbb{P}}$ -almost surely,  $\tilde{u}(\cdot, t)$  is strictly positive for almost all  $t \in [0, T_{max}]$ .

**Lemma 3.6.** *Let  $u_h \in U_h$  be strictly positive and let  $1 > \gamma \geq \frac{2}{p} + \frac{\varepsilon}{2} + \frac{\rho}{2p}$  and let*

$$\mathcal{E}_h(u_h) \leq C h^{-\rho/(2+p)}.$$

*Then, there exists an  $h$ -independent constant  $C_{osc} > 0$  such that the estimate*

$$\frac{u_h(x_i, y_j)}{u_h(x_{\hat{i}}, y_{\hat{j}})} \leq C_{osc} \quad (3.10)$$

holds true for all  $i \in \{1, \dots, \dim U_h^x\}$ ,  $j \in \{1, \dots, \dim U_h^y\}$ ,  $\hat{i} \in \{i-1, i, i+1\}$ , and  $\hat{j} \in \{j-1, j, j+1\}$ .

*Proof.* Using the standard embedding theorems for Hölder continuous functions and the discrete embedding proven in Corollary A.4, we obtain

$$\|u_h\|_{C^\gamma(\overline{\mathcal{O}})} \leq C \|\nabla u_h\|_{L^q(\mathcal{O})} \leq C \left( \|u_h\|_{H^1(\mathcal{O})} + \|\Delta_h u_h\|_{L^2(\mathcal{O})} \right) \leq C \sqrt{h^{-\varepsilon} \mathcal{E}_h(u_h)}$$

with  $q < \infty$  large enough. Furthermore, we have

$$\sup_{(x,y) \in \mathcal{O}} u_h^{-1} = \left( \sup_{(x,y) \in \mathcal{O}} u_h^{-p} \right)^{1/p} \leq C \left( h^{-2} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{F(u_h)\} dx dy \right)^{1/p} \leq C h^{-2/p} \mathcal{E}_h(u_h)^{1/p}. \quad (3.11)$$

Since there exists an element  $Q \in \mathcal{Q}_h$  including the vertices  $(x_i, y_j)$  and  $(x_{\hat{i}}, y_{\hat{j}})$  by assumption, we combine the estimates above and obtain

$$\begin{aligned} \left| \frac{u_h(x_i, y_j)}{u_h(x_{\hat{i}}, y_{\hat{j}})} - 1 \right| &= \left| \frac{u(x_i, y_j) - u_h(x_{\hat{i}}, y_{\hat{j}})}{u_h(x_{\hat{i}}, y_{\hat{j}})} \right| \leq C \sup_{(x,y) \in \mathcal{O}} u_h^{-1} h^\gamma \|u_h\|_{C^{0,\gamma}(\overline{\mathcal{O}})} \\ &\leq C h^{-2/p} \mathcal{E}_h(u_h)^{1/p} h^\gamma h^{-\varepsilon/2} \mathcal{E}_h(u_h)^{1/2} \\ &= C h^{-2/p - \varepsilon/2 + \gamma} \mathcal{E}_h(u_h)^{1/p + 1/2} \leq C =: C_{\text{osc}}, \end{aligned} \quad (3.12)$$

which completes the proof.  $\square$

We will start analyzing scheme (3.6) by showing that it admits a solution.

**Lemma 3.7.** *Let  $T_{\max}$  be a positive real number and  $\mathcal{E}_{\max,h} = \hat{C} h^{-\rho/(2+p)}$ . Then there exist stochastic processes  $u_h \in L^2(\Omega; C([0, T_{\max}]; U_h))$  and  $p_h \in L^2(\Omega; L^\infty(0, T_{\max}; U_h))$  as well as associated stopping times  $T_h$  such that:*

- *Almost surely, we have  $T_h = T_{\max} \wedge \inf \{t \in [0, \infty) : \mathcal{E}_h(u_h(\cdot, t)) \geq \mathcal{E}_{\max,h}\}$ .*
- *Almost surely, the process  $p_h$  solves (3.6b) for  $t \leq T_{\max}$  and is contained in  $C([0, T_h]; U_h)$ .*
- *Almost surely, the process  $u_h$  solves (3.6a) for  $t \leq T_{\max}$  and is constant for  $t \in [T_h, T_{\max}]$ .*

*Proof.* As the additional regularization term changes neither the Lipschitz continuity of  $\mathcal{E}_h(u_h)$  w.r.t.  $u_h$  when  $\mathcal{E}_h(u) \leq 2\mathcal{E}_{\max,h}$  nor the Lipschitz continuous dependence of  $p_h$  on  $u_h$  when  $\mathcal{E}_h(u_h) \leq 3\mathcal{E}_{\max,h}$ , the result follows along the lines of proof of Lemma 4.2 in [28].  $\square$

As the solutions  $u_h$  are continuous in space and time for  $h > 0$ , the positivity of the initial data immediately provides the positivity of the semidiscrete solutions.

**Corollary 3.8.** *The solutions constructed in Lemma 3.7 are strictly positive for all  $h > 0$ .*

#### 4. A PRIORI ESTIMATES

In this section, we shall establish uniform a priori estimates for the semi-discrete solution established in Lemma 3.7. These results will be used in the next section to pass to the limit  $h \searrow 0$  and this way to prove Theorem 3.5.

**4.1. The combined energy-entropy estimate.** We start this section by demonstrating that our spatial semidiscretization (3.6) satisfies a combined energy-entropy estimate as long as the energy remains below the critical threshold energy  $\mathcal{E}_{\max, h}$  which becomes infinite for  $h \searrow 0$ . Due to the cut-off mechanism implemented in (3.6), it is possible to extend the results to  $[0, T_{\max}]$ .

Writing  $u_h(x, y, t)$  as  $\sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} u_{ij}(t) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y)$  and choosing  $\psi_h(x, y) = \mathbf{e}_i^x(x) \mathbf{e}_j^y(y)$  in (3.6a) provides

$$\begin{aligned} & du_{ij} + \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x (\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \right\} dx dy dt \\ & + \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y (\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \right\} dx dy dt \\ & - \chi_{T_h} M_{ij}^{-1} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h, \text{loc}}^x \left\{ \partial_x (u_h \tilde{\mathbf{g}}_{h, kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy d\beta_{kl}^x \\ & - \chi_{T_h} M_{ij}^{-1} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^y \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h, \text{loc}}^y \left\{ \partial_y (u_h \tilde{\mathbf{g}}_{h, kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy d\beta_{kl}^y \end{aligned} \quad (4.1)$$

with  $M_{ij} = \int_{\mathcal{O}} \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) dx dy$ . As we assume the subdivision to be equidistant, we have  $M_{ij} = h_x h_y$  for all  $i \in \{1, \dots, \dim U_h^x\}$  and  $j \in \{1, \dots, \dim U_h^y\}$ .

Furthermore, we define for  $i \in \{1, \dots, \dim U_h^x\}$  and  $j \in \{1, \dots, \dim U_h^y\}$

$$\begin{aligned} L_{ij}(t) &:= -\chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x (\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \right\} dx dy \\ &\quad - \chi_{T_h} M_{ij}^{-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y (\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \right\} dx dy, \end{aligned} \quad (4.2)$$

$$Z_{ij}^x(\omega) := \chi_{T_h} M_{ij}^{-1} \sum_{k, l \in \mathbb{Z}} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h, \text{loc}}^x \left\{ \partial_x (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h, kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy, \quad (4.3)$$

$$Z_{ij}^y(\omega) := \chi_{T_h} M_{ij}^{-1} \sum_{k, l \in \mathbb{Z}} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h, \text{loc}}^y \left\{ \partial_y (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h, kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy. \quad (4.4)$$

Here,  $\langle \cdot, \cdot \rangle_{L^2}$  denotes the standard  $L^2(\mathcal{O})$  inner product. With this notation we may rewrite (4.1) as

$$du_{ij} = L_{ij}(t) dt + \sum_{k \in I_h^x, l \in I_h^y} (Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) d\beta_{kl}^x + Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl}) d\beta_{kl}^y). \quad (4.5)$$

For given positive parameters  $\alpha$  and  $\kappa$ , we consider the integral quantity

$$R(t) := R(\alpha, \kappa, h, u_h(t)) := \alpha + \mathcal{E}_h(u_h(t)) + \kappa \mathcal{S}_h(u_h(t)). \quad (4.6)$$

**Lemma 4.1.** *Let  $\bar{p} \geq 1$  be given. The first and second variations of  $R(s)^{\bar{p}}$  are given by*

$$D(R(s)^{\bar{p}}) = \bar{p} R(s)^{\bar{p}-1} (D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))) \quad (4.7a)$$

and

$$\begin{aligned} D^2(R(s)^{\bar{p}}) &= \bar{p} R(s)^{\bar{p}-1} (D^2\mathcal{E}_h(u_h(s)) + \kappa D^2\mathcal{S}_h(u_h(s))) \\ &\quad + \bar{p}(\bar{p} - 1) R(s)^{\bar{p}-2} (D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))) \otimes (D\mathcal{E}_h(u_h(s)) + \kappa D\mathcal{S}_h(u_h(s))) \end{aligned} \quad (4.7b)$$

with

$$D\mathcal{E}_h(u_h(s))\psi_h = \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \partial_x \psi_h \} + \mathcal{I}_h^x \{ \partial_y u_h \partial_y \psi_h \} dx dy \quad (4.8)$$

$$+ \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F'(u_h) \psi_h \} dx dy + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h u_h \Delta_h \psi_h \} dx dy, \quad (4.9)$$

$$D^2\mathcal{E}_h(u_h(s))(\phi_h, \psi_h) = \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \phi_h \partial_x \psi_h \} + \mathcal{I}_h^x \{ \partial_y \phi_h \partial_y \psi_h \} dx dy \\ + \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F''(u_h) \phi_h \psi_h \} dx dy + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h \phi_h \Delta_h \psi_h \} dx dy, \quad (4.10)$$

$$D\mathcal{S}_h(u_h(s))\psi_h = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ G'(u_h) \psi_h \} dx dy, \quad (4.11)$$

$$D^2\mathcal{S}_h(u_h(s))(\phi_h, \psi_h) = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ G''(u_h) \phi_h \psi_h \} dx dy. \quad (4.12)$$

Applying Itô's formula, we are able to show the following combined energy-entropy estimate.

**Proposition 4.2.** *Let  $\bar{p} \geq 1$  be arbitrary and let  $(u_h, p_h)$  be a solution to (3.6) for a parameter  $h \in (0, 1)$ . Furthermore, let the Assumptions **(B)**, **(B $\mathfrak{S}^*$ )**, **(I)**, **(P)**, **(R)**, and **(S)** hold true. Then, for sufficiently large  $\alpha$  and  $\kappa$  depending only on  $(\lambda_{kl}^x)_{kl}$ ,  $(\lambda_{kl}^y)_{kl}$ ,  $\bar{p}$ , and  $T_{max}$ , there exists a positive,  $h$ -independent constant  $C$  such that*

$$\mathbb{E} \left[ \sup_{t \in [0, T_{max}]} R(t)^{\bar{p}} \right] + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy ds \right] \\ + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} |\partial_y u_h|^2 \right\} dx dy ds \right] \\ + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \right] + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ |\partial_x \Delta_h u_h|^2 \right\} dx dy ds \right] \\ + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ |\partial_y \Delta_h u_h|^2 \right\} dx dy ds \right] \\ + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \right\} dx dy ds \right] \\ + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \right\} dx dy ds \right] \leq C. \quad (4.13)$$

*Proof.* Using the notation

$$\varphi_h(t) := \varphi_h(x, y, t) := \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} L_{ij}(t) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \quad (4.14)$$

and

$$\Phi_h(t)((\omega_x, \omega_y)^T) := \Phi_h(x, y, t)((\omega_x, \omega_y)^T) := \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} (Z_{ij}^x(\omega_x) + Z_{ij}^y(\omega_y)) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y), \quad (4.15)$$

we may rewrite (3.6) as

$$du_h = \varphi_h(t) dt + \Phi_h(t)(d\mathbf{W}_{Q,h}) \quad (4.16)$$

with

$$\mathbf{W}_{Q,h} := \sum_{\alpha \in \{x, y\}} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^\alpha \mathbf{g}_{kl} \beta_{kl}^\alpha \mathbf{b}_\alpha, \quad (4.17)$$

where  $\mathbf{b}_x, \mathbf{b}_y$  denote the standard Cartesian basis vectors of  $\mathbb{R}^2$ . Applying Itô's formula, we compute

$$\begin{aligned} R(t \wedge T_h)^{\bar{p}} &= R(0)^{\bar{p}} + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \varphi_h(s) ds \\ &\quad + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \Phi_h(s) d\mathbf{W}_{Q,h} \\ &\quad + \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} (D^2\mathcal{E}_h + \kappa D^2\mathcal{S}_h) (\Phi_{h,kl}^x, \Phi_{h,kl}^x) ds \\ &\quad + \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} (D^2\mathcal{E}_h + \kappa D^2\mathcal{S}_h) (\Phi_{h,kl}^y, \Phi_{h,kl}^y) ds \\ &\quad + \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}(\bar{p}-1)R(s)^{\bar{p}-2} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h) (\Phi_{h,kl}^x, \Phi_{h,kl}^x) ds \\ &\quad + \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}(\bar{p}-1)R(s)^{\bar{p}-2} (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h) (\Phi_{h,kl}^y, \Phi_{h,kl}^y) ds \\ &=: R(0)^{\bar{p}} + I + II + III + IV + V + VI, \end{aligned} \quad (4.18)$$

where we used the abbreviations

$$\Phi_{h,kl}^x(s) := \Phi_h(s)((\lambda_{kl}^x \mathbf{g}_{kl}, 0)^T) = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \quad (4.19a)$$

$$\text{and } \Phi_{h,kl}^y(s) := \Phi_h(s)((0, \lambda_{kl}^y \mathbf{g}_{kl})^T) = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y). \quad (4.19b)$$

Using Lemma 4.1, we compute

$$I = \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy L_{ij}(s) ds$$

$$\begin{aligned}
& + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \partial_y u_h \partial_y \mathbf{e}_j^y(y) \mathbf{e}_i^x(x) \} dx dy L_{ij}(s) ds \\
& + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F'(u_h) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy L_{ij}(s) ds \\
& + h^\varepsilon \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h u_h \Delta_h (\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \} dx dy L_{ij}(s) ds \\
& + \kappa \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ G'(u_h) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy L_{ij}(s) ds \\
& =: I_a + I_b + I_c + I_d + I_e. \tag{4.20}
\end{aligned}$$

From (4.2) we obtain by straightforward computations

$$\begin{aligned}
& \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ w \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} L_{ij}(s) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \\
& = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} M_{ij} L_{ij} w((i-1)h_x, (j-1)h_y) \\
& = -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G'''(u_h)]_x^{-1} \partial_x p_h \partial_x \mathcal{I}_h^{xy} \{ w \} \right\} + \mathcal{I}_h^y \left\{ [G'''(u_h)]_y^{-1} \partial_y p_h \partial_y \mathcal{I}_h^{xy} \{ w \} \right\} dx dy \tag{4.21}
\end{aligned}$$

for  $w \in C_{\text{per}}(\bar{\mathcal{O}})$  and therefore after integration by parts and using (3.6b)

$$\begin{aligned}
& I_a + I_b + I_c + I_d \\
& = \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (-\Delta_h u_h + F'(u_h) + h^\varepsilon \Delta_h \Delta_h u_h) \\
& \quad \times L_{ij}(s) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy ds \\
& = \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ p_h L_{ij}(s) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy ds \\
& = -\bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G'''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} + \mathcal{I}_h^y \left\{ [G'''(u_h)]_y^{-1} |\partial_y p_h|^2 \right\} dx dy ds. \tag{4.22}
\end{aligned}$$

Similarly, we use (4.21) with  $w = G'(u_h)$  and (2.16) to compute

$$\begin{aligned}
& \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} L_{ij}(s) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \\
&= -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x \mathcal{I}_h^{xy} \{G'(u_h)\} \right\} \\
&\quad + \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} \partial_y u_h \partial_y \mathcal{I}_h^{xy} \{G'(u_h)\} \right\} dx dy \\
&= -\chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \partial_x p_h \} + \mathcal{I}_h^x \{ \partial_y u_h \partial_y p_h \} dx dy \\
&= \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h u_h p_h \} dx dy \\
&= -\chi_{T_h} \|\Delta_h u_h\|_h^2 - \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \partial_x \mathcal{I}_h^{xy} \{F'(u_h)\} \} dx dy \\
&\quad + \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \partial_y u_h \partial_y \mathcal{I}_h^{xy} \{F'(u_h)\} \} dx dy + \chi_{T_h} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h u_h \Delta_h \Delta_h u_h \} dx dy \\
&= -\chi_{T_h} \|\Delta_h u_h\|_h^2 - \chi_{T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [F''(u_h)]_x |\partial_x u_h|^2 \} + \mathcal{I}_h^x \{ [F''(u_h)]_y |\partial_y u_h|^2 \} dx dy \\
&\quad - \chi_{T_h} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} + \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy.
\end{aligned} \tag{4.23}$$

From Assumption **(P)**, we obtain the estimate  $[F''(u_h)]_x \geq \tilde{c}_1 [|u_h|^{-p-2}]_x - \tilde{c}_2$  and  $[F''(u_h)]_y \geq \tilde{c}_1 [|u_h|^{-p-2}]_y - \tilde{c}_2$ . Therefore, combining the above estimates we conclude

$$\begin{aligned}
I_e &\leq -\kappa \bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\
&\quad - \kappa \bar{p} h^\varepsilon \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} + \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy \\
&\quad - \tilde{c}_1 \kappa \bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \} + \mathcal{I}_h^x \{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \} dx dy ds \\
&\quad + \tilde{c}_2 \kappa \bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds.
\end{aligned} \tag{4.24}$$

Noting

$$\begin{aligned}
D^2\mathcal{E}_h(\Phi_{h,kl}^x, \Phi_{h,kl}^x) &= \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy \\
&+ \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \partial_y \mathbf{e}_j^y(y) \right|^2 \right\} dx dy \\
&+ \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ F''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \\
&+ h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \left| \Delta_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy, \\
D^2\mathcal{S}_h(\Phi_{h,kl}^x, \Phi_{h,kl}^x) &= \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy, \quad (4.25)
\end{aligned}$$

and similar identities for  $D^2\mathcal{E}_h(\Phi_{h,kl}^y, \Phi_{h,kl}^y)$  and  $D^2\mathcal{S}_h(\Phi_{h,kl}^y, \Phi_{h,kl}^y)$ , we combine *III* and *IV* and obtain after reordering

$$\begin{aligned}
&\frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy ds \\
&+ \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \partial_y \mathbf{e}_j^y(y) \mathbf{e}_i^x(x) \right|^2 \right\} dx dy ds \\
&+ \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy ds \\
&+ \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl}) \partial_y \mathbf{e}_j^y(y) \mathbf{e}_i^x(x) \right|^2 \right\} dx dy ds \\
&+ \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ F''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy ds \\
&+ \frac{1}{2} \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ F''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy ds \\
&+ \frac{1}{2} h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \left| \Delta_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \left| \Delta_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^y (\lambda_{kl}^y \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy ds \\
& + \frac{1}{2} \kappa \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy ds \\
& + \frac{1}{2} \kappa \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G''(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} |Z_{ij}^y (\lambda_{kl}^y \mathbf{g}_{kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy ds \\
& = III_a + III_b + III_c + III_d + III_e + III_f + III_g + III_h + III_i + III_j. \tag{4.27}
\end{aligned}$$

To derive an estimate for  $III_a$ , we adapt the ideas of [28]. Using the periodicity, the specific form of the one dimensional stiffness matrix on equidistant meshes and  $M_{ij} = h_x h_y$  for all  $i = 1, \dots, \dim U_h^x$  and  $j = 1, \dots, \dim U_h^y$ , we compute using (2.13a) and (2.13b)

$$\begin{aligned}
& \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy \\
& = \int_{\mathcal{O}^y} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}^x} \left| \sum_{i=1}^{\dim U_h^x} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \right|^2 dx \mathbf{e}_j^y(y) dy \\
& = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}^y} (2(Z_{ij}^x)^2 - Z_{i-1,j}^x Z_{ij}^x - Z_{ij}^x Z_{i+1,j}^x) (\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_j^y(y) dy h_x^{-1} \\
& = \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \int_{\mathcal{O}^y} (Z_{i+1,j}^x - Z_{ij}^x)^2 (\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_j^y(y) dy h_x^{-1} \\
& = \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (\lambda_{kl}^x u_h \tilde{\mathbf{g}}_{h,kl}) \frac{\mathbf{e}_{i+1}^x(x) - \mathbf{e}_i^x(x)}{h_x} \right\} \mathbf{e}_j^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
& \leq 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x u_h \mathcal{I}_h^x \left\{ \tilde{\mathbf{g}}_{h,kl} \frac{\mathbf{e}_{i+1}^x(x) - \mathbf{e}_i^x(x)}{h_x} \right\} \mathbf{e}_j^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
& \quad + 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \left\{ u_h \frac{\mathbf{e}_{i+1}^x(x) - \mathbf{e}_i^x(x)}{h_x} \right\} \mathbf{e}_j^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
& = (*). \tag{4.28}
\end{aligned}$$

Recalling  $\mathbf{e}_{i+1}^x(x) = \mathbf{e}_i^x(x - h_x)$  and performing a discrete integration by parts (cf. Lemma A.6 in the appendix), we continue with the estimate

$$\begin{aligned}
(*) &\leq \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \partial_x^{-h_x} \tilde{\mathbf{g}}_{h,kl} \mathbf{e}_{i+1}^x(x) \} \mathbf{e}_j^y(y) \} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x^{+h_x} \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathbf{g}}_{h,kl} \mathbf{e}_i^x(x) \} \mathbf{e}_j^y(y) \} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \{ \partial_x^{-h_x} u_h \mathbf{e}_{i+1}^x(x) \} \mathbf{e}_j^y(y) \} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&+ \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x^{+h_x} \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \mathbf{e}_i^x(x) \} \mathbf{e}_j^y(y) \} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\leq \chi_{T_h} C \lambda_{kl}^{x^2} \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy \\
&+ \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\Delta_h^x u_h|^2 \} dx dy \\
&+ \chi_{T_h} C \lambda_{kl}^{x^2} \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\partial_x^{-h_x} u_h|^2 \} dx dy \\
&+ \chi_{T_h} C \lambda_{kl}^{x^2} \|\partial_x^{+h_x} \partial_x \tilde{\mathbf{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ u_h^2 \} dx dy \\
&\leq \chi_{T_h} C \lambda_{kl}^{x^2} \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 R(s) + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h^x u_h|^2 \} dx dy.
\end{aligned} \tag{4.29}$$

In the last step, we used Poincaré's inequality and the pathwise conservation of  $\int_{\mathcal{O}} u_h dx dy$  (see Remark 3.3, Assumption **(I)**, and the norm equivalence (2.6)), as well as

$$\begin{aligned}
\|\partial_x^{+h_x} \partial_x \mathcal{I}_h^x \{ \mathbf{g}_{kl} \}\|_{L^\infty(\mathcal{O})} &= \|\partial_x \mathcal{I}_h^x \{ \partial_x^{+h_x} \mathbf{g}_{kl} \}\|_{L^\infty(\mathcal{O})} \leq \|\partial_x \partial_x^{+h_x} \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})} \\
&\leq \|\partial_x \partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})} \leq \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}, \tag{4.30}
\end{aligned}$$

which follows from **(S)** and the stability of the nodal interpolation operator. This provides

$$\begin{aligned}
III_a &\leq C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\
&+ C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h^x u_h|^2 \} dx dy ds.
\end{aligned} \tag{4.31}$$

Similar computations show

$$\begin{aligned}
III_d &\leq C\bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^y{}^2 \|\mathfrak{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\
&\quad + C\bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^y{}^2 \|\mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^y u_h|^2\} dx dy ds.
\end{aligned} \tag{4.32}$$

To control  $III_b$ , we compute

$$\begin{aligned}
&\int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \left| \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \partial_y \mathbf{e}_j^y(y) \mathbf{e}_i^x(x) \right|^2 \right\} dx dy \\
&= \int_{\mathcal{O}^x} \mathcal{I}_h^x \left\{ \sum_{i=1}^{\dim U_h^x} \int_{\mathcal{O}^y} \left| \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathfrak{g}_{kl}) \partial_y \mathbf{e}_j^y(y) \right|^2 dy \mathbf{e}_i^x(x) \right\} dx \\
&= \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} (Z_{i,j+1}^x - Z_{i,j}^x)^2 (\lambda_{kl}^x \mathfrak{g}_{kl}) h_x h_y^{-1} \\
&= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_h^x \left\{ \partial_x (\lambda_{kl}^x \tilde{\mathfrak{g}}_{h,kl} u_h) \mathbf{e}_i^x(x) \frac{\mathbf{e}_{j+1}^y(y) - \mathbf{e}_j^y(y)}{h_y} \right\} \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\leq 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{u_h \mathbf{e}_i^x(x)\} \frac{\mathbf{e}_{j+1}^y(y) - \mathbf{e}_j^y(y)}{h_y} \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\quad + 2\chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathfrak{g}}_{h,kl} \mathbf{e}_i^x(x) \} \frac{\mathbf{e}_{j+1}^y(y) - \mathbf{e}_j^y(y)}{h_y} \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\leq \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x u_h \mathcal{I}_h^x \{ \partial_y^{-h_y} \tilde{\mathfrak{g}}_{h,kl} \mathbf{e}_i^x(x) \} \mathbf{e}_{j+1}^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_y^{+h_y} \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathfrak{g}}_{h,kl} \mathbf{e}_i^x(x) \} \mathbf{e}_j^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ \partial_y^{-h_y} u_h \mathbf{e}_i^x(x) \} \mathbf{e}_{j+1}^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\quad + \chi_{T_h} C \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \left( \lambda_{kl}^x \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_y^{+h_y} \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \mathbf{e}_i^x(x) \} \mathbf{e}_j^y(y) \right\} dx dy \right)^2 h_x^{-1} h_y^{-1} \\
&\leq \chi_{T_h} C \lambda_{kl}^x{}^2 \|\mathfrak{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x u_h|^2\} dx dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^x{}^2 \|\mathfrak{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_y^{+h_y} \partial_x u_h|^2\} dx dy
\end{aligned}$$

$$\begin{aligned}
& + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\partial_y^{-h_y} u_h|^2 \} dx dy \\
& + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 \} dx dy, \tag{4.33}
\end{aligned}$$

where we again used (2.13a) and (2.13b). Noting

$$\begin{aligned}
& \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_y^{+h_y} \partial_x u_h|^2 \} dx dy = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h^x u_h \Delta_h^y u_h \} dy dy \\
& \leq \frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h^x u_h|^2 \} dx dy + \frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h^y u_h|^2 \} dx dy = \frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy, \tag{4.34}
\end{aligned}$$

we obtain

$$\begin{aligned}
III_b & \leq C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\
& + C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy ds \tag{4.35}
\end{aligned}$$

and analogously

$$\begin{aligned}
III_c & \leq C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{y^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\
& + C \bar{p} \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^{y^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy ds \tag{4.36}
\end{aligned}$$

We shall apply a similar strategy to deal with  $III_g$  and  $III_h$ . We start with the decomposition

$$\begin{aligned}
& \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \left| \Delta_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right|^2 \right\} dx dy \\
& \leq 2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \sum_{j=1}^{\dim U_h^y} \left| \sum_{i=1}^{\dim U_h^x} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \Delta_h^x \mathbf{e}_i^x(x) \right|^2 \mathbf{e}_j^y(y) \right\} dx dy \\
& + 2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \sum_{i=1}^{\dim U_h^x} \left| \sum_{j=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \Delta_h^y \mathbf{e}_j^y(y) \right|^2 \mathbf{e}_i^x(x) \right\} dx dy \\
& =: III_{g_1} + III_{g_2}. \tag{4.37}
\end{aligned}$$

In order to control the first term, we use  $\mathbf{e}_{i-1}^x(x) = \mathbf{e}_i^x(x + h_x)$  and  $\mathbf{e}_{i+1}^x(x) = \mathbf{e}_i^x(x - h_x)$ , apply Lemma A.6, and compute using (2.13a) and (2.13b)

$$\sum_{i=1}^{\dim U_h^y} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) \Delta_h^x \mathbf{e}_i^x(x) = \sum_{i=1}^{\dim U_h^x} Z_{ij}^x (\lambda_{kl}^x \mathbf{g}_{kl}) h_x^{-2} (\mathbf{e}_{i-1}^x(x) - 2\mathbf{e}_i^x(x) + \mathbf{e}_{i+1}^x(x))$$

$$\begin{aligned}
&= \sum_{i=1}^{\dim U_h^x} h_x^{-2} \mathbf{e}_i^x(x) (-2Z_{ij}^x + Z_{i-1,j}^x + Z_{i+1,j}^x) (\lambda_{kl}^x \mathbf{g}_{kl}) \\
&= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \left\{ u_h \frac{\mathbf{e}_{i+1}^x(x) - 2\mathbf{e}_i^x(x) + \mathbf{e}_{i-1}^x(x)}{h_x^2} \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x u_h \mathcal{I}_h^x \left\{ \tilde{\mathbf{g}}_{h,kl} \frac{\mathbf{e}_{i+1}^x(x) - 2\mathbf{e}_i^x(x) + \mathbf{e}_{i-1}^x(x)}{h_x^2} \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&= \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \Delta_h^x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \left\{ u_h(x - h_x, y) \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \tilde{\mathbf{g}}_{h,kl}(x + h_x, y) \mathcal{I}_h^x \left\{ \Delta_h^x u_h \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} 2 \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \partial_x^{+h_x} \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \left\{ \partial_x^{-h_x} u_h \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \Delta_h^x u_h \mathcal{I}_h^x \left\{ \tilde{\mathbf{g}}_{h,kl}(x - h_x, y) \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x u_h(x + h_x, y) \mathcal{I}_h^x \left\{ \Delta_h^x \tilde{\mathbf{g}}_{h,kl} \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy \\
&+ \chi_{T_h} 2 \sum_{i=1}^{\dim U_h^x} \mathbf{e}_i^x(x) h_x^{-1} h_y^{-1} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \lambda_{kl}^x \partial_x \partial_x^{+h_x} u_h \mathcal{I}_h^x \left\{ \partial_x^{-h_x} \tilde{\mathbf{g}}_{h,kl} \mathbf{e}_i^x(x) \right\} \mathbf{e}_j^y(y) \right\} dx dy. \quad (4.38)
\end{aligned}$$

Similar to (4.30), we use **(S)** and the stability of the nodal interpolation operator to compute

$$\begin{aligned}
\|\partial_x \Delta_h^x \mathcal{I}_h^{xy} \{\mathbf{g}_{kl}\}\|_{L^\infty(\mathcal{O})} &= \|\partial_x \mathcal{I}_h^{xy} \{\partial_x^{+h_x} \partial_x^{-h_x} \mathbf{g}_{kl}\}\|_{L^\infty(\mathcal{O})} \\
&\leq \|\partial_x \partial_x^{+h_x} \partial_x^{-h_x} \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})} \leq \|\partial_x \partial_x \partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})} \leq \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}. \quad (4.39)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
III_{g_1} &\leq \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^x \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^x u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x \Delta_h^x u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^x u_h|^2\} \, dx \, dy.
\end{aligned} \tag{4.40}$$

Similar computations show

$$\begin{aligned}
III_{g_2} &\leq \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h^y u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^x \{|\partial_y u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x \Delta_h^y u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x u_h|^2\} \, dx \, dy \\
&\quad + \chi_{T_h} C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{\Delta_h^x u_h \Delta_h^y u_h\} \, dx \, dy.
\end{aligned} \tag{4.41}$$

Therefore,

$$\begin{aligned}
III_g &\leq C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}} \, ds \\
&\quad + C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h u_h|^2\} \, dx \, dy \, ds \\
&\quad + C \lambda_{kl}^{x^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{|\partial_x \Delta_h u_h|^2\} \, dx \, dy \, ds
\end{aligned} \tag{4.42}$$

holds true. Analogous computations provide

$$\begin{aligned}
III_h &\leq C\lambda_{kl}^{y^2} \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}} ds \\
&\quad + C\lambda_{kl}^{y^2} \|\mathbf{g}_{kl}\|_{W^{1,\infty}(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{|\Delta_h u_h|^2\} dx dy ds \\
&\quad + C\lambda_{kl}^{y^2} \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{|\partial_y \Delta_h u_h|^2\} dx dy ds.
\end{aligned} \tag{4.43}$$

To control  $III_e + III_f$  we compute for all  $i \in \{1, \dots, \dim U_h^x\}$ ,  $j \in \{1, \dots, \dim U_h^y\}$ ,  $k \in I_h^x$ , and  $l \in I_h^y$

$$\begin{aligned}
&\int_{\Omega} \mathcal{I}_h^{xy} \{F''(u_h) (|Z_{ij}^x(\lambda_{kl}^x \mathbf{g}_{kl})|^2 + |Z_{ij}^y(\lambda_{kl}^y \mathbf{g}_{kl})|^2) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y)\} dx dy \\
&= \chi_{T_h} F''(u_{ij}) M_{ij}^{-1} \left( \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \lambda_{kl}^x \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} \} dx dy \right)^2 \right. \\
&\quad \left. + \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \lambda_{kl}^y \partial_y (u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} \} dx dy \right)^2 \right) \\
&=: \chi_{T_h} F''(u_{ij}) M_{ij}^{-1} \left( \lambda_{kl}^{x^2} A_{ijkl}^2 + \lambda_{kl}^{y^2} B_{ijkl}^2 \right).
\end{aligned} \tag{4.44}$$

Using the definition of  $\mathcal{I}_{h,\text{loc}}^x$ , we obtain for the first term

$$\begin{aligned}
A_{ijkl}^2 &= \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} \} dx dy \right)^2 \\
&= \left( \sum_{Q \in \mathcal{Q}_h} \int_Q \mathcal{I}_h^{xy} \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \right)^2 \\
&\leq M_{ij} \sum_{Q \in \mathcal{Q}_h} \int_Q \mathcal{I}_h^{xy} \{ |\partial_x (u_h \tilde{\mathbf{g}}_{h,kl})|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \\
&\leq 2M_{ij} \sum_{Q \in \mathcal{Q}_h} \int_Q \mathcal{I}_h^{xy} \{ (|\partial_x u_h|^2 |\tilde{\mathbf{g}}_{h,kl}|^2 + |u_h|^2 |\partial_x \tilde{\mathbf{g}}_{h,kl}|^2) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \\
&\leq C \|\tilde{\mathbf{g}}_{h,k}\|_{W^{1,\infty}(\mathcal{O})} M_{ij} \sum_{Q \in \mathcal{Q}_h} \int_Q \mathcal{I}_h^{xy} \{ |\partial_x u_h|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \\
&\quad + C \|\tilde{\mathbf{g}}_{h,k}\|_{W^{1,\infty}(\mathcal{O})} M_{ij} \sum_{Q \in \mathcal{Q}_h} \int_Q \mathcal{I}_h^{xy} \{ |u_h|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \\
&= C \|\tilde{\mathbf{g}}_{h,kl}\|_{W^{1,\infty}(\mathcal{O})} M_{ij} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy \\
&\quad + C \|\tilde{\mathbf{g}}_{h,kl}\|_{W^{1,\infty}(\mathcal{O})} M_{ij} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} dx dy.
\end{aligned} \tag{4.45}$$

Similar computations for  $B_{ijkl}$  provide

$$\begin{aligned} III_e + III_f &\leq C \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ F''(u_h) \} |\partial_x u_h|^2 \} + \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ F''(u_h) \} |\partial_y u_h|^2 \} dx dy ds \\ &\quad + C \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 F''(u_h) \} dx dy ds. \end{aligned} \quad (4.46)$$

Analogously, we compute

$$\begin{aligned} III_i + III_j &\leq C\kappa \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} + \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ G''(u_h) \} |\partial_y u_h|^2 \} dx dy ds \\ &\quad + C\kappa \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 G''(u_h) \} dx dy ds. \end{aligned} \quad (4.47)$$

Collecting the above estimates and applying Assumptions **(B3)** and **(B3\*)**, we obtain

$$\begin{aligned} III + IV &\leq C \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\ &\quad + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy ds \\ &\quad + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy ds \\ &\quad + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ F''(u_h) \} |\partial_x u_h|^2 \} dx dy ds \\ &\quad + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ F''(u_h) \} |\partial_y u_h|^2 \} dx dy ds \\ &\quad + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 F''(u_h) \} dx dy ds \\ &\quad + C\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} dx dy ds \\ &\quad + C\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ G''(u_h) \} |\partial_y u_h|^2 \} dx dy ds \\ &\quad + C\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 G''(u_h) \} dx dy ds. \end{aligned} \quad (4.48)$$

While the first term on the right-hand side is a Gronwall term, the remaining terms need to be absorbed in the negative terms provided by  $I$ . For this reason, we need the following estimates: Due to Assumption **(P)**, we have  $F''(u_h) \leq C u_h^{-p-2} + C$ . Recalling the oscillation lemma 3.6, we obtain the estimates

$$\int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ F''(u_h) \} |\partial_x u_h|^2 \} dx dy \leq C \int_{\mathcal{O}} \mathcal{I}_h^y \{ [ |u_h|^{-p-2} ]_x |\partial_x u_h|^2 \} dx dy + CR(s), \quad (4.49)$$

$$\int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ F''(u_h) \} |\partial_y u_h|^2 \} dx dy \leq C \int_{\mathcal{O}} \mathcal{I}_h^x \{ [ |u_h|^{-p-2} ]_y |\partial_y u_h|^2 \} dx dy + CR(s). \quad (4.50)$$

Furthermore, Assumption **(P)**, Poincaré's inequality and our uniform control of the mass of discrete solutions (see Remark 3.3, Assumption **(I)**, and the norm equivalence (2.6))

provide

$$\begin{aligned} & \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 F''(u_h) \} dx dy \\ & \leq C \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(u_h) \} dx dy + C \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy + C \leq CR(s) \end{aligned} \quad (4.51)$$

for  $\alpha > 0$ . Applying Young's inequality and the Lemma 3.6, we compute

$$\begin{aligned} & \kappa \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} dx dy \\ & \leq \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ |u_h|^{-p-2} \} |\partial_x u_h|^2 \} dx dy + C\kappa^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy \\ & \leq C \int_{\mathcal{O}} \mathcal{I}_h^y \{ [ |u_h|^{-p-2} ]_x |\partial_x u_h|^2 \} dx dy + C\kappa^2 R(s) \end{aligned} \quad (4.52)$$

and

$$\kappa \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ G''(u_h) \} |\partial_y u_h|^2 \} dx dy \leq C \int_{\mathcal{O}} \mathcal{I}_h^x \{ [ |u_h|^{-p-2} ]_y |\partial_y u_h|^2 \} dx dy + C\kappa^2 R(s). \quad (4.53)$$

Noting the definition of  $G''(u_h)$ , we obtain

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 G''(u_h) dx dy \} \leq \int_{\mathcal{O}} 1 dx dy \leq C. \quad (4.54)$$

Therefore, the last term in (4.48) can be controlled by  $C\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} ds$  for  $\alpha > 0$ . This allows us to rewrite (4.48) for  $\kappa > 1$  as

$$\begin{aligned} III + IV & \leq C\kappa^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\ & + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy ds \\ & + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy ds \\ & + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [ |u_h|^{-p-2} ]_x |\partial_x u_h|^2 \} dx dy ds \\ & + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [ |u_h|^{-p-2} ]_y |\partial_y u_h|^2 \} dx dy ds. \end{aligned} \quad (4.55)$$

Using (3.6b), we compute

$$\begin{aligned}
& (D\mathcal{E}_h + \kappa D\mathcal{S}_h) \otimes (D\mathcal{E}_h + \kappa D\mathcal{S}_h)(\Phi_{h,kl}^x, \Phi_{h,kl}^x) \\
& \leq 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \partial_x u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \partial_x \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \right. \\
& \quad + \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \partial_y u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \mathbf{e}_i^x(x) \partial_y \mathbf{e}_j^y(y) \right\} dx dy \\
& \quad + \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ F'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \\
& \quad + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ \Delta_h u_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \Delta_h(\mathbf{e}_i^x(x) \mathbf{e}_j^y(y)) \right\} dx dy \Big)^2 \\
& \quad + 2\kappa^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \right)^2 \tag{4.56} \\
& = 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ p_h \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \right)^2 \\
& \quad + 2\kappa^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ G'(u_h) \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} Z_{ij}^x(\lambda_{kl}^x \mathfrak{g}_{kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} dx dy \right)^2 \\
& = \chi_{T_h} 2\lambda_{kl}^x{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x(u_h \tilde{\mathfrak{g}}_{h,kl}) p_h \} \} dx dy \right)^2 \\
& \quad + \chi_{T_h} 2\kappa^2 \lambda_{kl}^x{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x(u_h \tilde{\mathfrak{g}}_{h,kl}) G'(u_h) \} \} dx dy \right)^2 \\
& =: \chi_{T_h} 2\lambda_{kl}^x{}^2 A_I + \chi_{T_h} 2\kappa^2 \lambda_{kl}^x{}^2 A_{II},
\end{aligned}$$

where we used (4.3) in the last step. To control the first term on the right-hand side of (4.56), we use (2.13a) and (2.13b) to compute

$$A_I \leq 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathfrak{g}}_{h,kl} p_h \} \} dx dy \right)^2 + 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h p_h \} \} \right)^2 =: A_{I_a} + A_{I_b}. \tag{4.57}$$

Recalling (3.6b), **(P)**, the integration by parts formula used in (4.38), Hölder's inequality, and the positivity of  $u_h$ , we obtain

$$\begin{aligned}
A_{I_a} &\leq C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathbf{g}}_{h,kl} \Delta_h u_h \} \} dx dy \right)^2 \\
&\quad + C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h| \mathcal{I}_h^x \{ |\tilde{\mathbf{g}}_{h,kl}| (u_h^{-p-1} + 1) \} \} dx dy \right)^2 \\
&\quad + C \left( h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathbf{g}}_{h,kl} \Delta_h \Delta_h u_h \} \} dx dy \right)^2 \\
&\leq C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy \right) \|\tilde{\mathbf{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2 \|\Delta_h u_h\|_h^2 \\
&\quad + C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h| \mathcal{I}_h^x \{ |\tilde{\mathbf{g}}_{h,kl}| u_h^{-p/2} u_h^{-p/2-1} \} \} dx dy \right)^2 \\
&\quad + C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h| \mathcal{I}_h^x \{ |\tilde{\mathbf{g}}_{h,kl}| \} \} dx dy \right)^2 \\
&\quad + C \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h^x u_h|^2 \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy \\
&\quad + C \|\partial_x \partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy \\
&\quad + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h^x u_h|^2 \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy \\
&\quad + C \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h^y u_h|^2 \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy \\
&\quad + C \|\partial_y \partial_y \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy \\
&\quad + C \|\partial_y \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h^x u_h \Delta_h^y u_h \} dx dy h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h u_h|^2 \} dx dy,
\end{aligned} \tag{4.58}$$

where we used  $\int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \partial_y^{+h_y} u_h|^2 \} dx dy = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h^x u_h \Delta_h^y u_h \} dx dy$ . Using a discrete version of Hölder's inequality, we obtain

$$\int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ \Delta_h^x u_h \Delta_h^y u_h \} dx dy \leq \|\Delta_h^x u_h\|_h \|\Delta_h^y u_h\|_h \leq \|\Delta_h u_h\|_h^2. \tag{4.59}$$

Therefore, we have

$$\begin{aligned}
A_{I_a} &\leq CR(s) \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|\Delta_h u_h\|_h^2 \\
&\quad + C \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy \right) \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ u_h^{-p} \} dx dy \right) \\
&\quad + C \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy \\
&\quad + C \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 R(s) \left( h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right)
\end{aligned}$$

$$\begin{aligned} &\leq CR(s) \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy \right. \\ &\quad \left. + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right). \end{aligned} \quad (4.60)$$

Using (3.6b), Hölder's inequality, **(P)**, the computations used in (4.38), Poincaré's inequality, and  $R(s) \geq \alpha$ , we obtain the estimate

$$\begin{aligned} A_{I_b} &\leq C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \Delta_h u_h \} \} dx dy \right)^2 + C \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \{ u_h F'(u_h) \} \} dx dy \right)^2 \\ &\quad + C \left( h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathbf{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \Delta_h \Delta_h u_h \} \} dx dy \right)^2 \\ &\leq C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 \|\Delta_h u_h\|_h^2 \\ &\quad + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ u_h^{-p} \} dx dy \right)^2 + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \left( \int_{\mathcal{O}} u_h dx dy \right)^2 \\ &\quad + Ch^\varepsilon \|\partial_x \partial_x \partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\quad + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \|\Delta_h^x u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\quad + C \|\partial_x \partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\quad + Ch^\varepsilon \|\partial_x \partial_y \partial_y \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\quad + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \|\Delta_h^y u_h\|_h^2 h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\quad + C \|\partial_x \partial_y \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y u_h|^2 \} dx dy h^\varepsilon \|\Delta_h u_h\|_h^2 \\ &\leq C \left( \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \right) R(s) (\|\Delta_h u_h\|_h^2 + R(s)). \end{aligned} \quad (4.61)$$

Here, we used that  $\int_{\mathcal{O}} u_h dx dy = \int_{\mathcal{O}} u_h^0 dx dy$  is uniformly bounded  $\mathbb{P}$ -almost surely (cf. Assumption **(I)** and Remark 3.3). Using (2.13a) and (2.13b) and  $G'(u) = u^{-1}$ , we obtain

$$A_{II} \leq C \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} dx dy + C \|\partial_x \mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2. \quad (4.62)$$

Therefore, we have

$$\begin{aligned} V &\leq C \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \left( \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \right) \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\ &\quad + C \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \left( \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 + h^\varepsilon \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \right) \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\ &\quad + C \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy ds \end{aligned}$$

$$\begin{aligned}
& + C \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x{}^2 \|\mathbf{g}_{kl}\|_{W^{2,\infty}(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy ds \\
& + C \kappa^2 \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x{}^2 \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} dx dy ds.
\end{aligned} \tag{4.63}$$

Recalling Assumptions **(B3)** and **(B3\*)** and Lemma 3.6 and mimicking the computations in (4.52), we obtain for  $\kappa > 1$

$$\begin{aligned}
V & \leq C \kappa^4 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \} dx dy ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \} dx dy ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy ds.
\end{aligned} \tag{4.64}$$

Analogously, we compute

$$\begin{aligned}
VI & \leq C \kappa^4 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \} dx dy ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \} dx dy ds \\
& + C \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy ds.
\end{aligned} \tag{4.65}$$

In order to obtain bounds for the expected value of stochastic integral, we rewrite

$$\begin{aligned}
II & = \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\lambda_{kl}^x \tilde{\mathbf{g}}_{h,kl} u_h) p_h \} \} dx dy d\beta_{kl}^x \\
& + \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\lambda_{kl}^y \tilde{\mathbf{g}}_{h,kl} u_h) p_h \} \} dx dy d\beta_{kl}^y \\
& + \kappa \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\lambda_{kl}^x \tilde{\mathbf{g}}_{h,kl} u_h) G'(u_h) \} \} dx dy d\beta_{kl}^x \\
& + \kappa \sum_{k \in I_h^x, l \in I_h^y} \int_0^{t \wedge T_h} \bar{p} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\lambda_{kl}^y \tilde{\mathbf{g}}_{h,kl} u_h) G'(u_h) \} \} dx dy d\beta_{kl}^y \\
& =: II_a + II_b + II_c + II_d
\end{aligned} \tag{4.66}$$

and apply the Burkholder-Davis-Gundy inequality. For this reason, we introduce the Hilbert-Schmidt operators

$$\mathfrak{T}_1(s)(\omega) := \chi_{T_h} R(s)^{\bar{p}-1} \sum_{k \in I_h^x, l \in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h,kl}) p_h \} \} dx dy, \quad (4.67a)$$

$$\mathfrak{T}_3(s)(\omega) := \chi_{T_h} \kappa R(s)^{\bar{p}-1} \sum_{k \in I_h^x, l \in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h,kl}) G'(u_h) \} \} dx dy, \quad (4.67b)$$

mapping  $Q_x^{1/2} L^2(\mathcal{O})$  onto  $\mathbb{R}$  and the Hilbert-Schmidt operators

$$\mathfrak{T}_2(s)(\omega) := \chi_{T_h} R(s)^{\bar{p}-1} \sum_{k \in I_h^x, l \in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h,kl}) p_h \} \} dx dy, \quad (4.67c)$$

$$\mathfrak{T}_4(s)(\omega) := \chi_{T_h} \kappa R(s)^{\bar{p}-1} \sum_{k \in I_h^x, l \in I_h^y} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\langle \mathbf{g}_{kl}, \omega \rangle_{L^2} u_h \tilde{\mathbf{g}}_{h,kl}) G'(u_h) \} \} dx dy \quad (4.67d)$$

mapping  $Q_y^{1/2} L^2(\mathcal{O})$  onto  $\mathbb{R}$ . Recalling (4.60) and (4.61), we obtain

$$\begin{aligned} & \left( \sum_{k, l \in \mathbb{Z}} |\mathfrak{T}_1(s)(Q_x^{1/2} \mathbf{g}_{kl})|^2 \right)^{1/2} \\ &= \chi_{T_h} R(s)^{\bar{p}-1} \left( \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) p_h \} \} dx dy \right)^2 \right)^{1/2} \\ &\leq \chi_{T_h} C R(s)^{\bar{p}-1} \left( \sum_{k \in I_h^x, l \in I_h^y} \lambda_{kl}^x \left( \|\mathbf{g}_{kl}\|_{L^\infty(\mathcal{O})}^2 + h^\varepsilon \|\mathbf{g}_{kl}\|_{W^{3,\infty}(\mathcal{O})}^2 \right) \right. \\ &\quad \left. \times \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right)^2 \right)^{1/2} \\ &\leq \chi_{T_h} C R(s)^{\bar{p}-1} \left( \|\Delta_h u_h\|_h^2 + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right). \end{aligned} \quad (4.68)$$

Similarly, we have

$$\begin{aligned} & \left( \sum_{k, l \in \mathbb{Z}} |\mathfrak{T}_2(s)(Q_y^{1/2} \mathbf{g}_{kl})|^2 \right)^{1/2} \\ &\leq \chi_{T_h} C R(s)^{\bar{p}-1} \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y u_h|^2 \mathcal{I}_h^y \{ u_h^{-p-2} \} \} dx dy \right) \\ &\quad + \chi_{T_h} C R(s)^{\bar{p}-1} \left( h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy + R(s) \right). \end{aligned} \quad (4.69)$$

Following the computations in (4.62), we obtain

$$\left( \sum_{k,l \in \mathbb{Z}} |\mathfrak{I}_3(s)(Q_x^{1/2} \mathfrak{g}_{kl})|^2 \right)^{1/2} \leq \chi_{T_h} C \kappa R(s)^{\bar{p}-1} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} dx dy + C \right), \quad (4.70)$$

$$\left( \sum_{k,l \in \mathbb{Z}} |\mathfrak{I}_4(s)(Q_y^{1/2} \mathfrak{g}_{kl})|^2 \right)^{1/2} \leq \chi_{T_h} C \kappa R(s)^{\bar{p}-1} \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ G''(u_h) \} |\partial_y u_h|^2 \} dx dy + C \right). \quad (4.71)$$

Therefore, the Burkholder-Davis-Gundy inequality yields together with Young's inequality and  $R(s) \geq \alpha$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} |II_a| \right] \\ & \leq C \mathbb{E} \left[ \left( \int_0^{T_{\max} \wedge T_h} R(s)^{2\bar{p}-2} \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy \right. \right. \right. \\ & \quad \left. \left. \left. + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right) ds \right)^{1/2} \right] \end{aligned} \quad (4.72)$$

$$\begin{aligned} & \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T_{\max} \wedge T_h]} R(t)^{\bar{p}} \right] \\ & \quad + C \alpha^{-1} \mathbb{E} \left[ \int_0^{T_{\max} \wedge T_h} R(s)^{\bar{p}-1} \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \mathcal{I}_h^x \{ u_h^{-p-2} \} \} dx dy \right. \right. \\ & \quad \left. \left. + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} dx dy + R(s) \right) ds \right], \end{aligned}$$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} |II_b| \right] \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T_{\max} \wedge T_h]} R(t)^{\bar{p}} \right] \\ & \quad + C \alpha^{-1} \mathbb{E} \left[ \int_0^{T_{\max} \wedge T_h} R(s)^{\bar{p}-1} \left( \|\Delta_h u_h\|_h^2 + \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y u_h|^2 \mathcal{I}_h^y \{ u_h^{-p-2} \} \} dx dy \right. \right. \\ & \quad \left. \left. + h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} dx dy + R(s) \right) ds \right], \end{aligned} \quad (4.73)$$

and

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} |II_c| + \sup_{t \in [0, T_{\max}]} |II_d| \right] \leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \in [0, T_{\max} \wedge T_h]} R(t)^{\bar{p}-1} \right] \\ & \quad + C \alpha^{-1} \kappa^2 \mathbb{E} \left[ \int_0^{T_{\max} \wedge T_h} R(s)^{\bar{p}-1} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_h^x \{ G''(u_h) \} |\partial_x u_h|^2 \} \right) \right] \\ & \quad + C \alpha^{-1} \kappa^2 \mathbb{E} \left[ \int_0^{T_{\max} \wedge T_h} R(s)^{\bar{p}-1} \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_h^y \{ G''(u_h) \} |\partial_y u_h|^2 \} dx dy + C \right) \right]. \end{aligned} \quad (4.74)$$

Collecting the above results, we obtain

$$\begin{aligned}
\mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} |II| \right] &\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} R(t)^{\bar{p}} \right] + C\alpha^{-1} \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 \, dx \, dy \, ds \right] \\
&\quad + C\alpha^{-1} \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + C\alpha^{-1} \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + C\alpha^{-1} \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + C\alpha^{-1} \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + C\alpha^{-1} \kappa^4 \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}} \, ds \right], \tag{4.75}
\end{aligned}$$

where we again used Lemma 3.6 and mimicked (4.52). Collecting the intermediate results established above, we obtain for  $\kappa$  sufficiently large

$$\begin{aligned}
&\mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} R(t)^{\bar{p}} \right] + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [G''(u_h)]_y^{-1} |\partial_y u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \|\Delta_h u_h\|_h^2 \, ds \right] + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \Delta_h u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y \Delta_h u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [|u_h|^{-p-2}]_x |\partial_x u_h|^2 \} \, dx \, dy \, ds \right] \\
&\quad + \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [|u_h|^{-p-2}]_y |\partial_y u_h|^2 \} \, dx \, dy \, ds \right] \\
&\leq \mathbb{E} \left[ R(0)^{\bar{p}} \right] + C\kappa^4 \mathbb{E} \left[ \int_0^{T_h} R(s)^{\bar{p}} \, ds \right]. \tag{4.76}
\end{aligned}$$

Applying Gronwall's lemma concludes the proof.  $\square$

**4.2. Hölder continuity in time.** For compactness in time, the first step is to establish uniform Hölder continuity for the stochastic integral. In particular, we will prove the following lemma.

**Lemma 4.3.** *Let  $T_{\max} > 0$ ,  $\bar{p} > 1$ ,  $\nu \in (0, \frac{1}{2})$  and let the Assumptions  $(\mathbf{S})$ ,  $(\mathbf{I})$ ,  $(\mathbf{P})$ ,  $(\mathbf{B})$ ,  $(\mathbf{R})$ , and  $(\mathbf{B3}^*)$  hold true. If  $2\nu\bar{p} > 1$  holds, for solutions  $(u_h, p_h)$  to (3.6) the stochastic*

integral

$$I_h(t) := \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} \sum_{k \in I_h^x, l \in I_h^y} I_{ijkl}(t) \mathbf{e}_i^x(\tilde{x}) \mathbf{e}_j^y(\tilde{y}) \quad (4.77)$$

with

$$\begin{aligned} I_{ijkl}(t) := & M_{ij}^{-1} \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\lambda_{kl}^x u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} \} dx dy d\beta_{kl}^x \\ & + M_{ij}^{-1} \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\lambda_{kl}^x u_h \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \} \} dx dy d\beta_{kl}^y \end{aligned} \quad (4.78)$$

is contained in  $L^{2\bar{p}}(\Omega; C^\beta([0, T_{\max}]; L^2(\mathcal{O})))$  with  $\beta := \nu - \frac{1}{2\bar{p}}$ .

*Proof.* According to Lemma 2.1 in [29] it suffices to show that

$$\begin{aligned} \hat{Z}^x(s)(\omega) := & \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} M_{ij}^{-1} \mathbf{e}_i^x(\tilde{x}) \mathbf{e}_j^y(\tilde{y}) \\ & \times \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (u_h \sum_{k \in I_h^x, l \in I_h^y} \langle \mathbf{g}_{kl}, \omega \rangle_{L^2} \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy \end{aligned} \quad (4.79)$$

and

$$\begin{aligned} \hat{Z}^y(s)(\omega) := & \chi_{T_h} \sum_{i=1}^{\dim U_h^x} \sum_{j=1}^{\dim U_h^y} M_{ij}^{-1} \mathbf{e}_i^x(\tilde{x}) \mathbf{e}_j^y(\tilde{y}) \\ & \times \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^y \left\{ \partial_y (u_h \sum_{k \in I_h^x, l \in I_h^y} \langle \mathbf{g}_{kl}, \omega \rangle_{L^2} \tilde{\mathbf{g}}_{h,kl}) \mathbf{e}_i^x(x) \mathbf{e}_j^y(y) \right\} \right\} dx dy \end{aligned} \quad (4.80)$$

are progressively measurable and contained in  $L^{2\bar{p}}(\Omega \times (0, T_{\max}); L_2(L^2(\mathcal{O}); L^2(\mathcal{O})))$  with a uniform bound in  $h$  to establish  $I_h \in L^{2\bar{p}}(\Omega; W^{\nu, 2\bar{p}}(0, T_{\max}; L^2(\mathcal{O})))$ . Then the continuous embedding

$$W^{\nu, 2\bar{p}}(0, T_{\max}; L^2(\Omega)) \hookrightarrow C^{\nu - \frac{1}{2\bar{p}}}([0, T_{\max}]; L^2(\mathcal{O})) \quad (4.81)$$

completes the proof. Recalling the computations from (4.45) and using **(B3)**, we immediately obtain the bounds

$$\left\| \hat{Z}^x \right\|_{L_2(L^2(\mathcal{O}); L^2(\mathcal{O}))}^2 \leq \chi_{T_h} C \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x u_h|^2 \} dx dy + C \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 \} dx dy, \quad (4.82)$$

$$\left\| \hat{Z}^y \right\|_{L_2(L^2(\mathcal{O}); L^2(\mathcal{O}))}^2 \leq \chi_{T_h} C \int_{\mathcal{O}} \mathcal{I}_h^x \{ |\partial_y u_h|^2 \} dx dy + C \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |u_h|^2 \} dx dy. \quad (4.83)$$

Progressive measurability is satisfied due to the pathwise continuity of the  $u_h$   $\mathbb{P}$ -almost surely. Hence, the result follows by Proposition 4.2.  $\square$

In order to show compactness in time, we shall use Lemma 4.3 to establish the Hölder continuity of  $u_h$  as a mapping from  $[0, T_{\max}]$  into appropriate Sobolev spaces.

**Lemma 4.4.** *Let  $T_{max} > 0$  and let the Assumptions  $(\mathbf{S})$ ,  $(\mathbf{I})$ ,  $(\mathbf{P})$ ,  $(\mathbf{B})$ ,  $(\mathbf{R})$ , and  $(\mathbf{B}\mathfrak{B}^*)$  hold true. Then, for  $\bar{p}$  sufficiently large, a solution  $u_h$  to (3.6) is uniformly bounded in  $L^\sigma(\Omega; C^{1/4}([0, T_{max}]; (H^1_{\text{per}}(\mathcal{O}))')$ ) for  $\sigma < 8/5$ , i.e.*

$$\mathbb{E} \left[ \left( \sup_{t_1, t_2 \in [0, T_{max}]} \frac{\|u_h(t_2) - u_h(t_1)\|_{(H^1_{\text{per}}(\mathcal{O}))'}}{|t_2 - t_1|^{1/4}} \right)^\sigma \right] \leq C. \quad (4.84)$$

*Proof.* Denoting the standard  $L^2$ -projection onto  $U_h$  by  $\mathcal{P}_{U_h}$ , we obtain

$$\begin{aligned} \|u_h(t_2) - u_h(t_1)\|_{(H^1_{\text{per}}(\mathcal{O}))'} &= \sup_{0 \neq \psi \in H^1_{\text{per}}(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} \left| \int_{\mathcal{O}} (u_h(t_2) - u_h(t_1)) \psi \, dx \, dy \right| \right) = \\ &\leq \sup_{0 \neq \psi \in H^1_{\text{per}}(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} \left| \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (u_h(t_2) - u_h(t_1)) \mathcal{P}_{U_h} \{ \psi \} \} \, dx \, dy \right| \right) \\ &+ \sup_{0 \neq \psi \in H^1_{\text{per}}(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} \left| \int_{\mathcal{O}} (I - \mathcal{I}_h^{xy}) \{ (u_h(t_2) - u_h(t_1)) \mathcal{P}_{U_h} \{ \psi \} \} \, dx \, dy \right| \right) \\ &=: \sup_{0 \neq \psi \in H^1_{\text{per}}(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} |I| \right) + \sup_{0 \neq \psi \in H^1_{\text{per}}(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} |II| \right). \end{aligned} \quad (4.85)$$

To derive an estimate for  $|I|$ , we start with the identity

$$\begin{aligned} &\int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (u_h(t_2) - u_h(t_1)) \phi_h \} \, dx \, dy + \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [G''(u_h)]_x^{-1} \partial_x p_h \partial_x \phi_h \} \, dx \, dy \, ds \\ &+ \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [G''(u_h)]_y^{-1} \partial_y p_h \partial_y \phi_h \} \, dx \, dy \, ds = \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (I_h(t_2) - I_h(t_1)) \phi_h \} \, dx \, dy \end{aligned} \quad (4.86)$$

resulting from (3.6a) for  $0 \leq t_1 < t_2$ . Using (4.86) with  $\phi_h = \mathcal{P}_{U_h} \{ \psi \}$  and Hölder's inequality, we deduce the estimate

$$\begin{aligned} |I| &\leq \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \} \, dx \, dy \, ds \right)^{1/2} \\ &\quad \times \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ [G''(u_h)]_x^{-1} |\partial_x \mathcal{P}_{U_h} \{ \psi \}|^2 \} \, dx \, dy \, ds \right)^{1/2} \\ &+ \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [G''(u_h)]_y^{-1} |\partial_y p_h|^2 \} \, dx \, dy \, ds \right)^{1/2} \\ &\quad \times \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ [G''(u_h)]_y^{-1} |\partial_y \mathcal{P}_{U_h} \{ \psi \}|^2 \} \, dx \, dy \, ds \right)^{1/2} \\ &+ C \|I_h(t_2) - I_h(t_1)\|_{L^2(\mathcal{O})} \|\mathcal{P}_{U_h} \{ \psi \}\|_{L^2(\mathcal{O})}. \end{aligned} \quad (4.87)$$

In order to derive bounds for the first term on the right-hand side, we use the estimate

$$\begin{aligned}
& \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} |\partial_x \mathcal{P}_{U_h} \{\psi\}|^2 \right\} dx dy ds \right)^{1/2} \\
& \leq C \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \|u_h\|_{L^\infty(\mathcal{O})}^2 \|\partial_x \mathcal{P}_{U_h} \{\psi\}\|_{L^2(\mathcal{O})}^2 ds \right)^{1/2} \\
& \leq C \left( (t_2 - t_1)^{3/4} \|u_h\|_{L^8(0, T_{\max}; L^\infty(\mathcal{O}))}^2 \right)^{1/2} \|\psi\|_{H^1(\mathcal{O})} \\
& \leq C (t_2 - t_1)^{3/8} \|\psi\|_{H^1(\mathcal{O})} \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{1/4} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3/4} \\
& \quad + C (t_2 - t_1)^{3/8} \|\psi\|_{H^1(\mathcal{O})} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}.
\end{aligned} \tag{4.88}$$

In the last step, we used the inequality

$$\|u_h\|_{L^8(0, T_{\max}; L^\infty(\mathcal{O}))} \leq C \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{1/4} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3/4} + C \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))},$$

which follows from the discrete Gagliardo–Nirenberg inequality proven in Corollary A.4. Using similar computations for the second term on the right-hand side of (4.87) and estimating the remaining term with help of Lemma 4.3, we obtain

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \neq \psi \in H_{\text{per}}^1(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} |I| \right)^\sigma \right] \\
& \leq C (t_2 - t_1)^{3\sigma/8} \mathbb{E} \left[ \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy ds \right)^{\sigma/2} \right. \\
& \quad \left. \times \left( \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{\sigma/4} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3\sigma/4} + \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^\sigma \right) \right] \\
& \quad + C (t_2 - t_1)^{3\sigma/8} \mathbb{E} \left[ \left( \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} |\partial_y p_h|^2 \right\} dx dy ds \right)^{\sigma/2} \right. \\
& \quad \left. \times \left( \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{\sigma/4} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3\sigma/4} + \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^\sigma \right) \right] \\
& \quad + C (t_2 - t_1)^{\sigma\beta}.
\end{aligned} \tag{4.89}$$

Applying Young's inequality, we end up with

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \neq \psi \in H_{\text{per}}^1(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} |I| \right)^\sigma \right] \\
& \leq C (t_2 - t_1)^{3\sigma/8} \mathbb{E} \left[ \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy ds \right] \\
& \quad + C (t_2 - t_1)^{3\sigma/8} \mathbb{E} \left[ \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} |\partial_y p_h|^2 \right\} dx dy ds \right] \\
& \quad + C (t_2 - t_1)^{3\sigma/8} \mathbb{E} \left[ \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{\sigma/(4-2\sigma)} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3\sigma/(4-2\sigma)} + \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{2/(2-\sigma)} \right] \\
& \quad + C (t_2 - t_1)^{\sigma\beta}.
\end{aligned} \tag{4.90}$$

As we assumed  $\sigma < 8/5$ , we have  $\sigma/(4-2\sigma) < 2$ . Therefore, we may apply Young's inequality once again to obtain

$$\begin{aligned} \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{\sigma/(4-2\sigma)} \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{3\sigma/(4-2\sigma)} \\ \leq C \|\Delta_h u_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^2 + C \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{6\sigma/(8-5\sigma)}. \end{aligned} \quad (4.91)$$

In view of Proposition 4.2, we conclude

$$\mathbb{E} \left[ \sup_{0 \neq \psi \in H_{\text{per}}^1(\mathcal{O})} \left( \|\psi\|_{H^1(\mathcal{O})}^{-1} |I| \right)^\sigma \right] \leq C(t_2 - t_1)^{3\sigma/8} + C(t_2 - t_1)^{\sigma\beta} \quad (4.92)$$

for  $\bar{p}$  large enough.

Using (A.1e), an inverse estimate, and the stability properties of  $\mathcal{P}_{U_h}$ , we obtain

$$|II| \leq Ch \|u_h(t_2) - u_h(t_1)\|_{L^2(\mathcal{O})} \|\psi\|_{H^1(\mathcal{O})}. \quad (4.93)$$

To obtain an estimate for  $\|u_h(t_2) - u_h(t_1)\|_{L^2(\mathcal{O})}$ , we again start with the identity (4.86) and choose  $\phi_h = (u_h(t_2) - u_h(t_1))$ . This provides the estimate

$$\begin{aligned} \|u_h(t_2) - u_h(t_1)\|_h^2 &\leq \left| \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G'''(u_h)]_x^{-1} \partial_x p_h \partial_x (u_h(t_2) - u_h(t_1)) \right\} dx dy ds \right| \\ &\quad + \left| \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G'''(u_h)]_y^{-1} \partial_y p_h \partial_y (u_h(t_2) - u_h(t_1)) \right\} dx dy ds \right| \\ &\quad + \|I_h(t_2) - I_h(t_1)\|_{L^2(\mathcal{O})} \|u_h(t_2) - u_h(t_1)\|_{L^2(\mathcal{O})} \\ &=: N_1 + N_2 + N_3. \end{aligned} \quad (4.94)$$

Using Hölder's inequality and Young's inequality, we compute

$$\begin{aligned} N_1 &\leq \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| [G'''(u_h)]_x^{-1} \partial_x p_h \right|^{4/3} \right\} dx dy \right)^{3/4} \\ &\quad \times \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ |\partial_x (u_h(t_2) - u_h(t_1))|^4 \right\} dx dy \right)^{1/4} ds \\ &\leq C(t_2 - t_1)^{1/2} \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| [G'''(u_h)]_x^{-1} \partial_x p_h \right|^{4/3} \right\} dx dy \right)^{3/2} ds \\ &\quad + C(t_2 - t_1)^{-1/2} \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ |\partial_x (u_h(t_2) - u_h(t_1))|^4 \right\} dx dy \right)^{1/2} ds \\ &=: (t_2 - t_1)^{1/2} N_{1_a} + (t_2 - t_1)^{-1/2} N_{1_b}. \end{aligned} \quad (4.95)$$

For the first term, we obtain from Hölder's inequality

$$\begin{aligned} N_{1_a} &\leq C \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G'''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy \right) \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| [G'''(u_h)]_x^{-1} \right|^2 \right\} dx dy \right)^{1/2} ds \\ &\leq C \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} R(s) \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G'''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy ds, \end{aligned} \quad (4.96)$$

where we used

$$\left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \left| [G''(u_h)]_x^{-1} \right|^2 \right\} dx dy \right)^{1/2} \leq C \left( \int_{\mathcal{O}} |u_h|^4 dx dy \right)^{1/2} \leq C \|u_h\|_{H^1(\mathcal{O})}^2 \leq CR(s). \quad (4.97)$$

Concerning  $N_{1_b}$ , we obtain from a discrete version of the Gagliardo–Nirenberg inequality (cf. Corollary A.4)

$$\begin{aligned} N_{1_b} &\leq C(t_2 - t_1) \|u_h(t_2) - u_h(t_1)\|_{W^{1,4}(\mathcal{O})}^2 \\ &\leq C(t_2 - t_1) \|\Delta_h u_h(t_2) - \Delta_h u_h(t_1)\|_h \|u_h(t_2) - u_h(t_1)\|_{H^1(\mathcal{O})} \\ &\quad + C(t_2 - t_1) \|u_h(t_2) - u_h(t_1)\|_{H^1(\mathcal{O})}^2. \end{aligned} \quad (4.98)$$

Estimates for  $N_2$  can be derived in a similar manner. To derive bounds for  $N_3$ , we use the results from Lemma 4.3 and obtain

$$\begin{aligned} \mathbb{E}[N_3] &\leq \delta \mathbb{E}[\|u_h(t_2) - u_h(t_1)\|_h^2] + C_\delta \mathbb{E}[\|I_h(t_2) - I_h(t_1)\|_{L^2(\mathcal{O})}^2] \\ &\leq \delta \mathbb{E}[\|u_h(t_2) - u_h(t_1)\|_h^2] + C_\delta (t_2 - t_1)^{2\beta}. \end{aligned} \quad (4.99)$$

as the first term on the right-hand side of (4.99) can be absorbed for  $\delta$  small enough, we obtain

$$\begin{aligned} &\mathbb{E} \left[ h^\sigma \|u_h(t_2) - u_h(t_1)\|_{L^2(\mathcal{O})}^\sigma \right] \\ &\leq C(t_2 - t_1)^{\sigma/4} h^\sigma \mathbb{E} \left[ \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} R(s) \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(u_h)]_x^{-1} |\partial_x p_h|^2 \right\} dx dy ds \right] \\ &\quad + C(t_2 - t_1)^{\sigma/4} h^\sigma \mathbb{E} \left[ \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} R(s) \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ [G''(u_h)]_y^{-1} |\partial_y p_h|^2 \right\} dx dy ds \right] \\ &\quad + C(t_2 - t_1)^{\sigma/4} h^\sigma \mathbb{E} \left[ \|\Delta_h u_h(t_2) - \Delta_h u_h(t_1)\|_h^{\sigma/2} \|u_h(t_2) - u_h(t_1)\|_{H^1(\mathcal{O})}^{\sigma/2} \right] \\ &\quad + C(t_2 - t_1)^{\sigma/4} h^\sigma \mathbb{E} \left[ \|u_h(t_2) - u_h(t_1)\|_{H^1(\mathcal{O})}^\sigma \right] + Ch^\sigma (t_2 - t_1)^{\sigma/4} + Ch^\sigma (t_2 - t_1)^{\sigma\beta} \\ &\leq Ch^\sigma (t_2 - t_1)^{\sigma/4} + C(t_2 - t_1)^{\sigma/4} \mathbb{E} \left[ \sup_{s \in [0, T_{\max}]} \|u_h(s)\|_{H^1(\mathcal{O})}^\sigma \right] + Ch^\sigma (t_2 - t_1)^{\sigma\beta}, \end{aligned} \quad (4.100)$$

where we used (2.9). Choosing  $\beta \geq 1/4$  (cf. Lemma 4.3) completes the proof.  $\square$

## 5. PASSAGE TO THE LIMIT

**5.1. Compactness.** As  $u_h$  is only strictly positive for  $h > 0$ , we lack  $h$ -independent bounds on the pressure  $p_h$ . Therefore, we consider the fluxes

$$J_h^x := \mathcal{I}_h^y \left\{ \sqrt{[G''(u_h)]_x^{-1}} \partial_x p_h \right\}, \quad J_h^y := \mathcal{I}_h^x \left\{ \sqrt{[G''(u_h)]_y^{-1}} \partial_y p_h \right\}, \quad (5.1)$$

which are uniformly bounded in  $L^2(\Omega; L^2(0, T_{\max}; L^2(\mathcal{O})))$ . Note that solutions  $(u_h, p_h)$  to (3.6) may be equivalently characterized by  $(u_h, \Delta_h u_h, J_h^x, J_h^y)$ . In the following, we consider these objects in the spaces

$$\mathcal{X}_u := C([0, T_{\max}]; L^q(\mathcal{O})), \quad q < \infty, \quad (5.2a)$$

$$\mathcal{X}_{\Delta u} := (L^2(0, T_{\max}; L^2(\mathcal{O})))_{weak}, \quad (5.2b)$$

$$\mathcal{X}_{J^x} := (L^2(0, T_{\max}; L^2(\mathcal{O})))_{weak}, \quad (5.2c)$$

$$\mathcal{X}_{J^y} := (L^2(0, T_{\max}; L^2(\mathcal{O})))_{weak}. \quad (5.2d)$$

**Lemma 5.1.** *Let  $T_{\max} > 0$  be arbitrary but fixed. Let  $(u_h, \Delta_h u_h, J_h^x, J_h^y)_h$  be a sequence of discrete solutions to (3.6). Then the families of laws  $(\mu_{u_h})_h$ ,  $(\mu_{\Delta_h u_h})_h$ ,  $(\mu_{J_h^x})_h$ , and  $(\mu_{J_h^y})_h$  are tight.*

*Proof.* From Proposition 4.2 and Lemma 4.4, we obtain that  $(u_h)_h$  is uniformly bounded in  $L^2(\Omega; L^\infty(0, T_{\max}; H^1(\mathcal{O}))) \cap L^\sigma(\Omega; C^{1/4}([0, T_{\max}]; (H_{\text{per}}^1(\mathcal{O}))'))$  for  $\sigma < 8/5$ . Due to the well-known compactness theorem by Simon (cf. [64]), the ball  $\overline{B}_R$  in  $L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O})) \cap C^{1/4}([0, T_{\max}]; (H_{\text{per}}^1(\mathcal{O}))')$  is a compact subset of  $C([0, T_{\max}]; L^q(\mathcal{O}))$ . Furthermore, we have for any  $R > 0$

$$\begin{aligned} \mu_{u_h}(\mathcal{X}_u \setminus \overline{B}_R) &= \mathbb{P} \left[ \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^\sigma + \|u_h\|_{C^{1/4}([0, T_{\max}]; (H_{\text{per}}^1(\mathcal{O}))')}^\sigma > R^\sigma \right] \\ &\leq R^{-\sigma} \mathbb{E} \left[ \|u_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^2 + C + \|u_h\|_{C^{1/4}([0, T_{\max}]; (H_{\text{per}}^1(\mathcal{O}))')}^\sigma \right], \end{aligned} \quad (5.3)$$

which shows the tightness of  $(\mu_{u_h})_h$ . As closed balls in  $L^2(0, T_{\max}; L^2(\mathcal{O}))$  are compact in the weak topology, the tightness of  $(\mu_{\Delta_h u_h})_h$ ,  $(\mu_{J_h^x})_h$  and  $(\mu_{J_h^y})_h$  is a direct consequence of Markov's inequality and the bound obtained in Proposition 4.2.  $\square$

Following the lines of [28], we introduce the Polish space

$$\mathcal{X}_{\mathbf{W}} := (C([0, T_{\max}]; L^2(\mathcal{O})))^2 \quad (5.4)$$

as an additional path space. Let  $\mu_{\mathbf{W}} := (\mu_{W^x}, \mu_{W^y})^T$  be the law of

$$\mathbf{W} = \left( \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^x \mathfrak{g}_{kl} \beta_{kl}^x, \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^y \mathfrak{g}_{kl} \beta_{kl}^y \right)^T. \quad (5.5)$$

As  $\mathcal{X}_{\mathbf{W}}$  is a Polish space,  $\mu_{\mathbf{W}}$  is a Radon measure and therefore regular from the interior, i.e.

$$\mu_{\mathbf{W}} \left( (C([0, T_{\max}]; L^2(\mathcal{O})))^2 \right) = \sup \left\{ \mu_{\mathbf{W}}(K) : K \subset (C([0, T_{\max}]; L^2(\mathcal{O})))^2 \text{ compact} \right\}. \quad (5.6)$$

To deal with the initial data, we introduce the space  $\mathcal{X}_{u_0} := H_{\text{per}}^1(\mathcal{O})$ . Together with the tightness results of Lemma 5.1, we obtain the following result.

**Lemma 5.2.** *On the path space  $\mathcal{X} := \mathcal{X}_u \times \mathcal{X}_{\Delta u} \times \mathcal{X}_{J^x} \times \mathcal{X}_{J^y} \times \mathcal{X}_{\mathbf{W}} \times \mathcal{X}_{u_0}$  the joint laws  $\mu_h$  defined by*

$$\begin{aligned} \mu_h(A \times B \times C \times D \times E \times F) &:= \\ \mathbb{P} \left[ \{u_h \in A\} \cap \{\Delta_h u_h \in B\} \cap \{J_h^x \in C\} \cap \{J_h^y \in D\} \cap \{\mathbf{W} \in E\} \cap \{u^0 \in F\} \right], \end{aligned} \quad (5.7)$$

for  $h \in (0, 1]$  are tight.

Using Jakubowski's theorem (cf. [51]) which is a generalization of Skorokhod's theorem (cf. [65]), we obtain the following result.

**Proposition 5.3.** *Let  $(u_h, \Delta_h u_h, J_h^x, J_h^y)$  be solutions to (3.6) in the sense of Lemma 3.7 defined on the same stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  with respect to the Wiener process  $\mathbf{W}$ . Then there exists a subsequence which we again denote by  $(u_h, \Delta_h u_h, J_h^x, J_h^y)$  such that there are a stochastic basis  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ , a sequence of random variables*

$$\tilde{u}_h : \tilde{\Omega} \rightarrow C([0, T_{\max}]; L^q(\mathcal{O})) \quad (q < \infty), \quad (5.8a)$$

$$\widetilde{\Delta_h u_h} : \tilde{\Omega} \rightarrow L^2(0, T_{\max}; L^2(\mathcal{O})), \quad (5.8b)$$

$$\tilde{J}_h^x : \tilde{\Omega} \rightarrow L^2(0, T_{\max}; L^2(\mathcal{O})), \quad (5.8c)$$

$$\tilde{J}_h^y : \tilde{\Omega} \rightarrow L^2(0, T_{\max}; L^2(\mathcal{O})), \quad (5.8d)$$

$$\tilde{u}_h^0 : \tilde{\Omega} \rightarrow H_{\text{per}}^1(\mathcal{O}), \quad (5.8e)$$

a sequence of  $(L^2(\mathcal{O}))^2$ -valued processes  $\tilde{\mathbf{W}}_h$  on  $\tilde{\Omega}$ , random variables

$$\tilde{u} \in L^2(\tilde{\Omega}; C([0, T_{\max}]; L^q(\mathcal{O}))) \quad (q < \infty), \quad (5.9a)$$

$$\widetilde{\Delta u} \in L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.9b)$$

$$\tilde{J}^x \in L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.9c)$$

$$\tilde{J}^y \in L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.9d)$$

$$\tilde{u}^0 \in L^2(\tilde{\Omega}; H_{\text{per}}^2(\mathcal{O})), \quad (5.9e)$$

and an  $L^2(\mathcal{O})$ -valued process  $\tilde{W}$  on  $\tilde{\Omega}$  which satisfy the following properties:

- i) The law of  $(\tilde{u}_h, \widetilde{\Delta_h u_h}, \tilde{J}_h^x, \tilde{J}_h^y, \tilde{W}_h, \tilde{u}_h^0)$  on  $\mathcal{X}_u \times \mathcal{X}_{\Delta u} \times \mathcal{X}_{J^x} \times \mathcal{X}_{J^y} \times \mathcal{X}_{\mathbf{W}} \times \mathcal{X}_{u_0}$  under  $\tilde{\mathbb{P}}$  coincides for any  $h$  with the law of  $(u_h, \Delta_h u_h, J_h^x, J_h^y, W, u_h^0)$  under  $\mathbb{P}$ .
- ii) The sequence  $(\tilde{u}_h, \widetilde{\Delta_h u_h}, \tilde{J}_h^x, \tilde{J}_h^y, \tilde{W}_h, \tilde{u}_h^0)$  converges  $\tilde{\mathbb{P}}$ -almost surely towards  $(\tilde{u}, \widetilde{\Delta u}, \tilde{J}^x, \tilde{J}^y, \tilde{W}, \tilde{u}^0)$  in the topology of  $\mathcal{X}$ .

**Remark 5.4.** *In particular, one may use the interval  $[0, 1]$  for  $\tilde{\Omega}$ , its standard Borel  $\sigma$ -algebra for  $\tilde{\mathcal{F}}$ , and the Lebesgue measure for  $\tilde{\mathbb{P}}$  (cf. [51]).*

Similarly to  $T_h$ , we introduce the random stopping times

$$\tilde{T}_h := T_{\max} \wedge \inf \{t \geq 0 : \mathcal{E}_h(\tilde{u}_h(t)) \geq \mathcal{E}_{\max, h}\}. \quad (5.10)$$

**Lemma 5.5.** *Along a subsequence, the convergence  $\lim_{h \searrow 0} \tilde{T}_h = T_{\max}$  holds  $\tilde{\mathbb{P}}$ -almost surely.*

*Proof.* Following the lines of [28], we compute for each  $\tau \in (0, T_{\max}]$

$$\tilde{\mathbb{P}}[\{\tilde{T}_h < \tau\}] = \mathbb{P}[\{T_h < \tau\}] \leq Ch^{\rho/(2+p)}. \quad (5.11)$$

Hence  $\tilde{T}_h \rightarrow T_{\max}$  in probability for  $h \searrow 0$ , which implies the  $\tilde{\mathbb{P}}$ -almost sure convergence for a subsequence.  $\square$

**Lemma 5.6.** *Under the Assumptions (S), (I), (P), (B), (R), and (B3\*),  $\widetilde{\Delta_h u_h}$  can be identified  $\tilde{\mathbb{P}}$ -almost surely as the discrete Laplacian of  $\tilde{u}_h$ , i.e.*

$$\widetilde{\Delta_h u_h} = \Delta_h \tilde{u}_h. \quad (5.12a)$$

Furthermore, the flux components  $\tilde{J}_h^x$  and  $\tilde{J}_h^y$  can be identified  $\tilde{\mathbb{P}}$ -almost surely as

$$\tilde{J}_h^x = \chi_{\tilde{T}_h} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x (-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy} \{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h) \right\}, \quad (5.12b)$$

$$\tilde{J}_h^y = \chi_{\tilde{T}_h} \mathcal{I}_h^x \left\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \partial_y (-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy} \{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h) \right\}. \quad (5.12c)$$

*Proof.* As  $\Delta_h u_h$  depends continuously on  $u_h$  (cf. (2.9)), (5.12a) follows by equality of laws. As for every fixed  $h > 0$ , the functions  $u_h$  and  $\tilde{u}_h$  are almost surely in  $C([0, T_{\max}]; C(\overline{\mathcal{O}}))$ , the stopping times  $T_h(\omega)$  and  $\tilde{T}_h(\tilde{\omega})$  are also continuous functions of  $u_h$  and  $\tilde{u}_h$ , respectively. By inverse estimates (cf. Theorem 4.5.11 in [10]) and the oscillation lemma 3.6, the same holds true for the terms on the right-hand side of (5.12b) and (5.12c). In particular, the expectation

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left| \int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{J}_h^x \phi \, dx \, dy \, dt \right. \right. \\ \left. \left. - \int_0^{T_{\max}} \chi_{\tilde{T}_h} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x (-\Delta_h \tilde{u}_h + \mathcal{I}_h^{xy} \{F'(\tilde{u}_h)\} + h^\varepsilon \Delta_h \Delta_h \tilde{u}_h) \right\} \right| \right] \end{aligned} \quad (5.13)$$

coincides with

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left| \int_0^{T_{\max}} \int_{\mathcal{O}} J_h^x \phi \, dx \, dy \, dt \right. \right. \\ \left. \left. - \int_0^{T_{\max}} \chi_{T_h} \mathcal{I}_h^y \left\{ \sqrt{[G''(u_h)]_x^{-1}} \partial_x (-\Delta_h u_h + \mathcal{I}_h^{xy} \{F'(u_h)\} + h^\varepsilon \Delta_h \Delta_h u_h) \right\} \right| \right] \end{aligned} \quad (5.14)$$

for arbitrary  $\phi \in C^\infty([0, T_{\max}]; (C_{\text{per}}^\infty(\overline{\mathcal{O}})))$ . As the latter one is equal to zero, this gives the claim w.r.t.  $\tilde{J}_h^x$ . The argumentation for  $\tilde{J}_h^y$  is the same.  $\square$

**Corollary 5.7.** *Let the assumptions of Proposition 5.3 and Lemma 5.6 hold true. Then the limit function  $\tilde{\Delta}u$  introduced in (5.8b) can be identified with the Laplacian of  $\tilde{u}$   $\tilde{\mathbb{P}}$ -almost surely.*

*Proof.* Choosing  $\phi \in C_{\text{per}}^\infty(\overline{\mathcal{O}})$ , a Taylor expansion provides  $\partial_x^{-h_x} \partial_x^{+h_x} \phi + \partial_y^{-h_y} \partial_y^{+h_y} \phi \rightarrow \Delta \phi$  in  $L^\infty(\mathcal{O})$ . Therefore, shifting the discrete Laplacian onto the test function before passing to the limit provides the desired result.  $\square$

We proceed by showing that  $\tilde{\mathbf{W}}$  and  $\tilde{\mathbf{W}}_h$  are  $Q$ -Wiener processes adapted to suitably defined filtrations  $(\tilde{\mathcal{F}}_t)_{t \geq 0}$  and  $(\tilde{\mathcal{F}}_{h,t})_{t \geq 0}$ , respectively. We define  $(\tilde{\mathcal{F}}_t)_{t \geq 0}$  to be the  $\tilde{\mathbb{P}}$ -augmented canonical filtration associated with  $(\tilde{u}, \tilde{\mathbf{W}}, \tilde{u}^0)$ , i.e.

$$\tilde{\mathcal{F}}_t := \sigma(\sigma(r_t \tilde{u}, r_t \tilde{\mathbf{W}}) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}^0)). \quad (5.15)$$

Here,  $r_t$  is the restriction of a function defined on  $[0, T_{\max}]$  to the interval  $[0, t]$  with  $t \in [0, T_{\max}]$ . Analogously, we introduce the filtrations  $(\tilde{\mathcal{F}}_{h,t})_{t \geq 0}$  as the  $\tilde{\mathbb{P}}$ -augmented canonical filtration associated with  $(\tilde{u}_h, \tilde{\mathbf{W}}_h, \tilde{u}_h^0)$

$$\tilde{\mathcal{F}}_{h,t} := \sigma(\sigma(r_t \tilde{u}_h, r_t \tilde{\mathbf{W}}_h) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}_h^0)). \quad (5.16)$$

**Lemma 5.8.** *The processes  $\tilde{\mathbf{W}}_h$  and  $\tilde{\mathbf{W}}$  are  $Q$ -Wiener processes adapted to the filtrations  $(\tilde{\mathcal{F}}_{h,t})_{t \geq 0}$  and  $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ . They can be written as*

$$\tilde{\mathbf{W}}_h(t) = \sum_{\alpha \in \{x,y\}} \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^\alpha \mathbf{g}_{kl} \tilde{\beta}_{h,kl}^\alpha \mathbf{b}_\alpha \quad \text{and} \quad \tilde{\mathbf{W}}(t) = \sum_{\alpha \in \{x,y\}} \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^\alpha \mathbf{g}_{kl} \tilde{\beta}_{kl}^\alpha \mathbf{b}_\alpha. \quad (5.17)$$

Here,  $(\tilde{\beta}_{h,kl}^\alpha)_{\alpha \in \{x,y\}, k,l \in \mathbb{Z}}$  and  $(\tilde{\beta}_{kl}^\alpha)_{\alpha \in \{x,y\}, k,l \in \mathbb{Z}}$  are families of independently and identically distributed Brownian motions with respect to  $(\tilde{\mathcal{F}}_{h,t})_{t \geq 0}$  and  $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ .

For a proof we refer to [28]. Combining the results of Proposition 5.3 and Lemma 5.6 with the discrete Gagliardo–Nirenberg inequality allows us to establish improved convergence results.

**Lemma 5.9.** *Let  $\tilde{u}_h$  and  $\Delta_h \tilde{u}_h$  be the random variables identified in Proposition 5.3. Furthermore, let the Assumptions **(S)**, **(I)**, **(P)**, **(B)**, **(R)**, and **(B $\mathfrak{S}^*$ )** hold true and let  $q \in (1, \infty)$ . Then  $\tilde{u}_h$  converges strongly along a subsequence towards  $\tilde{u}$  in  $L^2(0, T_{\max}; W^{1,q}(\mathcal{O}))$   $\tilde{\mathbb{P}}$ -almost surely.*

*Proof.* We follow the lines of Lemma 5.1 in [59]. Using Hölder’s inequality, we compute

$$\|\tilde{u}_h - \tilde{u}\|_{L^2(0, T_{\max}; W^{1,q}(\mathcal{O}))} \leq C \|\tilde{u}_h - \tilde{u}\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))}^{1/q} \|\tilde{u}_h - \tilde{u}\|_{L^2(0, T_{\max}; W^{1,2q-2}(\mathcal{O}))}^{(q-1)/q}. \quad (5.18)$$

Due to the discrete Gagliardo–Nirenberg inequality (cf. Lemma A.2), we have  $\tilde{u} \in L^2(0, T_{\max}; W_{\text{per}}^{1,q}(\mathcal{O}))$   $\tilde{\mathbb{P}}$ -almost surely. As  $\tilde{u}_h$  is  $\tilde{\mathbb{P}}$ -almost surely in  $U_h \cap L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O}))$  with  $\Delta_h \tilde{u}_h \in L^2(0, T_{\max}; L^2(\mathcal{O}))$ , we may use the discrete Gagliardo–Nirenberg inequality (cf. Corollary A.4) to show that  $\tilde{u}$  is also  $\tilde{\mathbb{P}}$ -almost surely in  $L^2(0, T_{\max}; W_{\text{per}}^{1,q}(\mathcal{O}))$ . Therefore, it suffices to show that  $\tilde{u}_h$  converges strongly towards  $\tilde{u}$  in  $L^2(0, T_{\max}; H^1(\mathcal{O}))$ . Using the triangle inequality, we derive

$$\|\tilde{u}_h - \tilde{u}\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))} \leq \|\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))} + \|\mathcal{R}\{\tilde{u}\} - \tilde{u}\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))}, \quad (5.19)$$

where  $\mathcal{R}$  is the Ritz projection operator defined in (2.17). As  $\mathcal{R}\{\tilde{u}\}$  converges strongly towards  $\tilde{u}$  in  $H^1(\mathcal{O})$  and since  $\tilde{u}_h$  is  $L^\infty(H^1)$ -regular, it only remains to show that the first term on the right-hand side vanishes. We define  $A_h : U_h \rightarrow U_h \cap H_*^1(\mathcal{O})$  via  $\int_{\mathcal{O}} (A_h \phi_h) \psi_h \, dx \, dy = \int_{\mathcal{O}} \nabla \phi_h \cdot \nabla \psi_h \, dx \, dy$  for all  $\phi_h, \psi_h \in U_h$  and compute

$$\begin{aligned} \|\nabla \tilde{u}_h - \nabla \mathcal{R}\{\tilde{u}\}\|_{L^2(0, T; L^2(\mathcal{O}))}^2 &\leq \left| \int_0^{T_{\max}} \int_{\mathcal{O}} (A_h(\tilde{u}_h - \mathcal{R}\{\tilde{u}\})) \cdot (\tilde{u}_h - \mathcal{R}\{\tilde{u}\}) \, dx \, dy \right| \\ &\leq \|A_h(\tilde{u}_h - \mathcal{R}\{\tilde{u}\})\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \|\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \\ &\leq C \|\tilde{u}_h - \mathcal{R}\{\tilde{u}\}\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}. \end{aligned} \quad (5.20)$$

Together with the strong convergence of  $\tilde{u}_h$  in  $L^2(0, T_{\max}; L^2(\mathcal{O}))$   $\tilde{\mathbb{P}}$ -almost surely, which we have from Proposition 5.3, and the strong convergence of  $\mathcal{R}\{\tilde{u}\}$  towards  $\tilde{u}$ , we complete the proof.  $\square$

**5.2. Convergence of the deterministic terms.** In this section, we identify the limit functions  $\tilde{J}_h^x$  and  $\tilde{J}_h^y$  introduced in Proposition 5.3 and use the a priori estimates from

Proposition 4.2 to establish additional (weak) convergence properties. In order to identify the limit of the fluxes, we consider the discrete pressure

$$U_h \ni \tilde{p}_h := -\chi_{\tilde{T}_h} \Delta_h \tilde{u}_h + \chi_{\tilde{T}_h} \mathcal{I}_h^{xy} \{F'(\tilde{u}_h)\} + \chi_{\tilde{T}_h} h^\varepsilon \Delta_h \Delta_h \tilde{u}_h$$

$\tilde{\mathbb{P}}$ -almost everywhere. In addition, we introduce the sets

$$S_\delta := \left\{ (\tilde{\omega}, t, (x, y)) \in \tilde{\Omega} \times (0, T_{\max}) \times \mathcal{O} : \tilde{u}(\tilde{\omega}, t, x, y) > \delta \right\}, \quad (5.21a)$$

$$S_\delta(\tilde{\omega}, t) := \{(x, y) \in \mathcal{O} : \tilde{u}(\tilde{\omega}, t, (x, y)) > \delta\}, \quad (5.21b)$$

$$S_\delta^{\mathcal{Q}_h} := \left\{ (\tilde{\omega}, t, (x, y)) \in \tilde{\Omega} \times (0, T_{\max}) \times \mathcal{O} : \exists Q \in \mathcal{Q}_h \text{ s.t. } (x, y) \in Q \right. \\ \left. \text{and } \tilde{u}_h(\tilde{\omega}, t, \cdot)|_Q > \delta \right\}, \quad (5.21c)$$

$$S_\delta^{\mathcal{Q}_h}(\tilde{\omega}, t) := \left\{ (x, y) \in \mathcal{O} : \exists Q \in \mathcal{Q}_h \text{ s.t. } (x, y) \in Q \text{ and } \tilde{u}_h(\tilde{\omega}, t, \cdot)|_Q > \delta \right\}. \quad (5.21d)$$

On these superlevel sets, we will be able to identify the limit functions of  $\tilde{p}_h$ ,  $\tilde{J}_h^x$ , and  $\tilde{J}_h^y$ . In particular, the following lemma holds true.

**Lemma 5.10.** *Let  $\tilde{u}_h$  and  $\tilde{u}$  be the random variables identified in Proposition 5.3. Furthermore, let the Assumptions **(S)**, **(I)**, **(P)**, **(B)**, **(R)**, and **(B3\*)** hold true. Then, there exists a subsequence, again denoted by  $\tilde{u}_h$  such that for all  $q < \infty$  and  $\bar{q}$  the following convergence properties hold true:*

$$\tilde{u}_h \rightarrow \tilde{u} \quad \text{in } L^q(\tilde{\Omega}; C([0, T_{\max}]; L^q(\mathcal{O}))), \quad (5.22a)$$

$$\tilde{u}_h \rightarrow \tilde{u} \quad \text{in } L^{\bar{q}}(\tilde{\Omega}; L^2(0, T_{\max}; W^{1,q}(\mathcal{O}))), \quad (5.22b)$$

$$\tilde{u}_h \xrightarrow{*} \tilde{u} \quad \text{in } L^{2\bar{p}}(\tilde{\Omega}; L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O}))), \quad (5.22c)$$

$$\Delta_h \tilde{u}_h \rightharpoonup \Delta \tilde{u} \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.22d)$$

$$\mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \rightarrow \tilde{u} \quad \text{in } L^q(\tilde{\Omega}; L^\infty(0, T_{\max}; L^q(\mathcal{O}))), \quad (5.22e)$$

$$\mathcal{I}_h^x \left\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \right\} \rightarrow \tilde{u} \quad \text{in } L^q(\tilde{\Omega}; L^\infty(0, T_{\max}; L^q(\mathcal{O}))) \quad (5.22f)$$

In addition, we have

$$\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \partial_x \tilde{p}_h \rightharpoonup \chi_{[S_\delta]} \partial_x \tilde{p} \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.23a)$$

$$\chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \partial_y \tilde{p}_h \rightharpoonup \chi_{[S_\delta]} \partial_y \tilde{p} \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.23b)$$

with  $\tilde{p} = -\Delta \tilde{u} + F'(\tilde{u})$  on  $S_\delta$  for all  $\delta > 0$  and

$$\tilde{J}_h^x \rightharpoonup \tilde{J}^x \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.24a)$$

$$\tilde{J}_h^y \rightharpoonup \tilde{J}^y \quad \text{in } L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.24b)$$

where  $\tilde{J}^x$  and  $\tilde{J}^y$  are the limit functions introduced in Proposition 5.3. For every  $\delta > 0$ , we are able to identify these limit functions on the superlevel sets  $S_\delta$  as  $\tilde{J}^x = \tilde{u} \partial_x (-\Delta \tilde{u} + F'(\tilde{u}))$  and  $\tilde{J}^y = \tilde{u} \partial_y (-\Delta \tilde{u} + F'(\tilde{u}))$ .

*Proof.* By Proposition 5.3, we have in particular  $\tilde{u}_h \rightarrow \tilde{u}$ . Choosing  $\bar{p}$  sufficiently large in Proposition 4.2 and combining Proposition 5.3 with Vitali's convergence theorem and with the bounds on  $\tilde{u}_h$  and on  $\Delta_h \tilde{u}_h$  (see Proposition 4.2), we obtain the strong convergence of  $\tilde{u}_h$  towards  $\tilde{u}$  in  $L^q(\tilde{\Omega}; C([0, T_{\max}]; L^q(\Omega)))$ , which gives (5.22a). To establish (5.22b), we use the continuous Gagliardo–Nirenberg inequality and Lemma A.2 to show that

$$\begin{aligned} \|\tilde{u}_h - \hat{u}\|_{L^2(0, T_{\max}; W^{1, q}(\mathcal{O}))}^{\hat{q}} &\leq C \|\Delta_h \tilde{u}_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}^{\hat{q}(q-2)/q} \|\tilde{u}_h\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{\hat{q}-\hat{q}(q-2)/q} \\ &\quad + C \|\tilde{u}\|_{L^2(0, T_{\max}; H^2(\mathcal{O}))}^{\hat{q}(q-2)/q} \|\tilde{u}\|_{L^\infty(0, T_{\max}; H^1(\mathcal{O}))}^{\hat{q}-\hat{q}(q-2)/q} \\ &\quad + C \|\tilde{u}_h\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))}^{\hat{q}} + C \|\tilde{u}\|_{L^2(0, T_{\max}; H^1(\mathcal{O}))}^{\hat{q}} \end{aligned} \quad (5.25)$$

for all  $\bar{q} < \hat{q} < \frac{2q}{q-2}$ . As this choice of  $\hat{q}$  in particular implies that  $\hat{q}(q-2)/q < 2$ , we may use Hölder's inequality to show that  $\tilde{\mathbb{E}}\left[\|\tilde{u}_h - \tilde{u}\|_{L^2(0, T_{\max}; W^{1, q}(\mathcal{O}))}^{\hat{q}}\right]$  is uniformly bounded.

As we already established the  $\tilde{\mathbb{P}}$ -almost sure convergence in Proposition 5.3, an application of Vitali's convergence theorem provides the result.

From the bounds on  $\tilde{u}_h$  stated in Proposition 4.2, we obtain the weak convergence  $\tilde{u}_h \rightharpoonup^* \mathbf{u}$  in  $L^{2\bar{p}}(\tilde{\Omega}; L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O})))$  along a subsequence. As the strong convergence of  $\tilde{u}_h$  towards  $\tilde{u}$  is already established in (5.22a), we are able to identify  $\mathbf{u}$  and  $\tilde{u}$ .

To establish the weak convergence expressed in (5.22d), we again start with the uniform  $L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$ -bounds on  $\Delta_h \tilde{u}_h$  stated in Proposition 5.3. These bounds provide the existence of a subsequence converging weakly in  $L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$  towards some limit function  $\mathbf{v}$ . As we also have  $\Delta_h \tilde{u}_h \rightharpoonup \Delta \tilde{u}$   $\tilde{\mathbb{P}}$ -almost surely (cf. Proposition 5.3 and Corollary 5.7), combining the aforementioned uniform bounds and Vitali's convergence theorem provides

$$\int_0^{T_{\max}} \int_{\mathcal{O}} \Delta_h \tilde{u}_h \phi \, dx \, dy \, dt \rightarrow \int_0^{T_{\max}} \int_{\mathcal{O}} \Delta \tilde{u} \phi \, dx \, dy \, dt \quad (5.26)$$

strongly in  $L^r(\tilde{\Omega})$  for  $r < 2$ . Therefore, we have  $\mathbf{v} = \Delta_h \tilde{u}$ .

To establish (5.22e), it suffices to show  $\tilde{\mathbb{E}}\left[\left\|\mathcal{I}_h^y \left\{\sqrt{[G''(\tilde{u}_h)]_x^{-1}}\right\} - \tilde{u}_h\right\|_{L^\infty(0, T_{\max}; L^q(\mathcal{O}))}^q\right]$  vanishes for  $h \searrow 0$ . Using Hölder's inequality, we obtain

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\sup_{t \in [0, T_{\max}]} \int_{\mathcal{O}} \left|\mathcal{I}_h^y \left\{\sqrt{[G''(\tilde{u}_h)]_x^{-1}}\right\} - \tilde{u}_h\right|^q \, dx \, dy\right] \\ &\leq \left(\tilde{\mathbb{E}}\left[\sup_{t \in [0, T_{\max}]} \int_{\mathcal{O}} \left|\mathcal{I}_h^y \left\{\sqrt{[G''(\tilde{u}_h)]_x^{-1}}\right\} - \tilde{u}_h\right|^2 \, dx \, dy\right]\right)^{1/2} \\ &\quad \times \left(\tilde{\mathbb{E}}\left[\sup_{t \in [0, T_{\max}]} \int_{\mathcal{O}} \left|\mathcal{I}_h^y \left\{\sqrt{[G''(\tilde{u}_h)]_x^{-1}}\right\} - \tilde{u}_h\right|^{2q-2} \, dx \, dy\right]\right)^{1/2}. \end{aligned} \quad (5.27)$$

In the following, we will show that the first factor converges towards zero while the second factor remains bounded. Since  $[G''(\tilde{u}_h)]_x^{-1}|_{K^x}(y) \in [\min_{x \in K^x} \tilde{u}_h^2(x, y), \max_{x \in K^x} \tilde{u}_h^2(x, y)]$ , we compute on each  $Q \in \mathcal{Q}_h$

$$\begin{aligned} \int_Q \left| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} - \tilde{u}_h \right|^2 dx dy &\leq \int_Q \mathcal{I}_h^y \left\{ \left| \max_{K^x} \tilde{u}_h - \min_{K^x} \tilde{u}_h \right|^2 \right\} dx dy \\ &\leq h^2 \int_Q \mathcal{I}_h^y \left\{ |\partial_x \tilde{u}_h|^2 \right\} dx dy, \end{aligned} \quad (5.28)$$

which provides the first result. To show that the second integral remains bounded, we use that on each element  $K^x = (ih_x, (i+1)h_x)$  we have  $[G''(\tilde{u}_h)]_x^{-1}|_{K^x}(y) = \tilde{u}_h(ih_x, y) \cdot \tilde{u}_h((i+1)h_x, y)$ . Applying the inequality of arithmetic and geometric means and Jensen's inequality, we obtain on each  $Q := (ih_x, (i+1)h_x) \times K^y$

$$\begin{aligned} \int_Q \left| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right|^{2q-2} dx dy &\leq \int_Q \left| \mathcal{I}_h^y \left\{ \sqrt{\tilde{u}_h(ih_x, y) \tilde{u}_h((i+1)h_x, y)} \right\} \right|^{2q-2} dx dy \\ &\leq \int_Q \left| \frac{1}{2} \mathcal{I}_h^y \left\{ \tilde{u}_h(ih_x, y) + \tilde{u}_h((i+1)h_x, y) \right\} \right|^{2q-2} dx dy \\ &\leq \int_Q \frac{1}{2} \mathcal{I}_h^y \left\{ \tilde{u}_h^{2q-2}(ih_x, y) + \tilde{u}_h^{2q-2}((i+1)h_x, y) \right\} dx dy = \int_Q \mathcal{I}_h^{xy} \left\{ \tilde{u}_h^{2q-2} \right\} dx dy. \end{aligned} \quad (5.29)$$

Summing over all  $Q \in \mathcal{Q}_h$  and applying (2.6), we obtain the result for  $\bar{p}$  large enough. The convergence expressed in (5.22f) can be shown with analogous computations.

To address (5.23a), we combine the bounds in Proposition 5.3 with the estimate

$$\int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{O}_h}]} \chi_{[S_\delta]} |\partial_x \tilde{p}_h|^2 dx dy \leq \delta^{-2} C \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ [G''(\tilde{u}_h)]_x^{-1} |\partial_x \tilde{p}_h|^2 \right\} dx dy, \quad (5.30)$$

which indicates that  $\chi_{[S_{\delta/4}^{\mathcal{O}_h}]} \chi_{[S_\delta]} \partial_x \tilde{p}_h$  is uniformly bounded in  $L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$ .

Therefore, there exists a subsequence converging towards a limit function  $\eta_\delta$ . In the following, we have to show that  $\eta_\delta = \partial_x \tilde{p}$  on  $S_\delta$ .

We approximate the characteristic function  $\chi_{[S_\delta]}$  by a family of functions  $\tilde{\chi}_N(\omega, t)_{N \in \mathbb{N}} : \tilde{\Omega} \times [0, T_{\max}] \rightarrow C_{\text{per}}^\infty(\mathcal{O})$  satisfying

$$\begin{aligned} \tilde{\chi}_N(\omega, t, (x, y)) &= 0 \quad \text{for } (x, y) \in S_\delta(\omega, t)^c, \\ \tilde{\chi}_N(\omega, t, (x, y)) &= \begin{cases} 1 & \text{if } \text{dist}((x, y), \partial S_\delta(\omega, t)) \geq \frac{1}{N}, \\ 0 & \text{if } \text{dist}((x, y), \partial S_\delta(\omega, t)) \leq \frac{1}{N+1}. \end{cases} \end{aligned} \quad (5.31)$$

To identify  $\eta_\delta$  on  $S_\delta$  with  $\partial_x \tilde{p}$ , we will show that

$$\tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \eta_\delta \tilde{\phi} \tilde{\chi}_N dx dy dt \right] = -\tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{p} \partial_x (\tilde{\phi} \tilde{\chi}_N) dx dy dt \right] \quad (5.32)$$

for sufficiently regular test functions  $\tilde{\phi}$ . In the following, we will show that this equation is valid even in the slightly more general case when  $\tilde{\phi} \tilde{\chi}$  is replaced by the test function  $\tilde{\zeta} \in L^\infty(\tilde{\Omega}; C^\infty([0, T_{\max}]; C_{\text{per}}^\infty(\mathcal{O})))$  with  $\text{supp } \tilde{\zeta}(\omega, t) \subset S_\delta(\omega, t)$ .

As  $\tilde{u}_h$  converges strongly in  $W_{\text{per}}^{1,q}(\mathcal{O})$  ( $q < \infty$ )  $\tilde{\mathbb{P}}$ -almost surely for almost all  $t \in (0, T_{\max})$  and hence almost everywhere in  $\tilde{\Omega} \times (0, T_{\max})$ , we have  $\tilde{u}_h \rightarrow \tilde{u}$  in  $C_{\text{per}}^\gamma(\mathcal{O})$  with  $\gamma < 1$  almost everywhere in  $\tilde{\Omega} \times (0, T_{\max})$ . This allows us to apply an appropriate version of Egorov's theorem (cf. Theorem 42 in [19]) and to deduce for all  $\iota > 0$  the existence of a

subset  $\mathfrak{E}_\iota \subset \tilde{\Omega} \times (0, T_{\max})$  with measure smaller than  $\iota$  such that  $\tilde{u}_h$  converges uniformly in  $(\tilde{\Omega} \times (0, T_{\max})) \setminus \mathfrak{E}_\iota =: \mathfrak{E}_\iota^c$ . Therefore, we compute

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \partial_x \tilde{p}_h \tilde{\zeta} \, dx \, dy \, dt \right] &= \int_{\tilde{\Omega}} \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \tilde{p}_h \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &= \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \tilde{p}_h \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] + \int_{\mathfrak{E}_\iota} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \partial_x \tilde{p}_h \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] =: A + B. \end{aligned} \quad (5.33)$$

Applying Hölder's inequality, we immediately obtain  $|B| \leq C\iota^{1/2}$ . As  $\tilde{u}_h$  converges uniformly towards  $\tilde{u}$  in  $\mathfrak{E}_\iota^c$ , we have  $\text{supp } \tilde{\zeta}(\omega, t) \subset S_\delta(\omega, t) \subset S_{\delta/4}^{\mathcal{Q}_h}(\omega, t)$  for  $h$  sufficiently small. Using the definition of  $\tilde{p}_h$ , we obtain

$$\begin{aligned} A &= - \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \tilde{p}_h \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &= \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} \Delta_h \tilde{u}_h \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] - \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} \mathcal{I}_h^{xy} \{F'(\tilde{u}_h)\} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} \Delta_h \Delta_h \tilde{u}_h \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &=: A_1 + A_2 + A_3. \end{aligned} \quad (5.34)$$

The convergence of  $A_1$  towards  $\int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \Delta \tilde{u} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}]$  is a direct consequence of the weak convergence (5.22d). To obtain the convergence of  $A_2$ , we use

$$A_2 = - \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} F'(\tilde{u}_h) \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] + \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} (I - \mathcal{I}_h^{xy}) \{F'(\tilde{u}_h)\} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}].$$

As  $\text{supp } \tilde{\zeta}(\omega, t) \subset S_{\delta/4}^{\mathcal{Q}_h}(\omega, t)$ , the second term vanishes for  $h \searrow 0$  due to Lemma A.5 and the first term converges towards  $-\int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} F'(\tilde{u}) \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}]$  due to Vitali's convergence theorem. The treatment of  $A_3$  is more delicate, as the bounds on  $h^\varepsilon \Delta_h \Delta_h \tilde{u}_h$  are not obvious. We begin by splitting  $A_3$  into

$$\begin{aligned} A_3 &= - h^\varepsilon \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} \mathcal{I}_h^{xy} \left\{ \Delta_h \Delta_h \tilde{u}_h \mathcal{I}_h^{xy} \left\{ \partial_x \tilde{\zeta} \right\} \right\} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} (I - \mathcal{I}_h^{xy}) \left\{ \Delta_h \Delta_h \tilde{u}_h \mathcal{I}_h^{xy} \left\{ \partial_x \tilde{\zeta} \right\} \right\} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &\quad - h^\varepsilon \int_{\mathfrak{E}_\iota^c} \int_{\mathcal{O}} \chi_{\tilde{T}_h} \Delta_h \Delta_h \tilde{u}_h (I - \mathcal{I}_h^{xy}) \left\{ \partial_x \tilde{\zeta} \right\} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] =: A_{3_a} + A_{3_b} + A_{3_c}. \end{aligned} \quad (5.35)$$

Recalling (2.7) and applying Hölder's inequality shows that  $A_{3_a}$  vanishes, since

$$\begin{aligned} |A_{3_a}| &\leq h^{\varepsilon/2} \left\| h^{\varepsilon/2} \partial_x \Delta_h \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \left\| \partial_x \mathcal{I}_h^{xy} \left\{ \partial_x \tilde{\zeta} \right\} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \\ &\quad + h^{\varepsilon/2} \left\| h^{\varepsilon/2} \partial_y \Delta_h \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \left\| \partial_y \mathcal{I}_h^{xy} \left\{ \partial_x \tilde{\zeta} \right\} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \\ &\leq h^{\varepsilon/2} C. \end{aligned} \quad (5.36)$$

Due to (2.9) we have  $h \left\| h^{\varepsilon/2} \Delta_h \Delta_h \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \leq C$ . Therefore, standard estimates for  $\mathcal{I}_h^{xy}$ , which can be found e.g. in Theorem 4.4.20 in [10], provide

$$\begin{aligned} |A_{3_b}| &\leq Ch^{\varepsilon/2} h \left\| h^{\varepsilon/2} \Delta_h \Delta_h \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \left\| \nabla \mathcal{I}_h^{xy} \left\{ \partial_x \tilde{\zeta} \right\} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \\ &\leq Ch^{\varepsilon/2}. \end{aligned} \quad (5.37)$$

Similar considerations based on Lemma A.1 provide

$$\begin{aligned} |A_{3_c}| &\leq h^{\varepsilon/2} \left\| h^{\varepsilon/2} \Delta_h \Delta_h \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \left\| (I - \mathcal{I}_h^{xy}) \left\{ \partial_x \tilde{\zeta} \right\} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \\ &\leq Ch^{1+\varepsilon/2} \left\| \partial_x \tilde{\zeta} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; H^2(\mathcal{O})))} . \end{aligned} \quad (5.38)$$

Collecting the results above, we have

$$\begin{aligned} A &\rightarrow \int_{\mathfrak{E}_t^c} \int_{\mathcal{O}} \Delta \tilde{u} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] - \int_{\mathfrak{E}_t^c} \int_{\mathcal{O}} F'(\tilde{u}) \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &= - \int_{\tilde{\Omega}} \int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{p} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] - \int_{\mathfrak{E}_t^c} \int_{\mathcal{O}} \Delta \tilde{u} \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] \\ &\quad + \int_{\mathfrak{E}_t^c} \int_{\mathcal{O}} F'(\tilde{u}) \partial_x \tilde{\zeta} \, dx \, dy \, dt \, d\tilde{\mathbb{P}}[\tilde{\omega}] . \end{aligned} \quad (5.39)$$

As  $\tilde{u} > \delta$  in  $\text{supp } \partial_x \tilde{\zeta}$ , the last two integrals can be bounded by  $C\iota^{1/2}$ . Therefore, we may identify  $\eta_\delta$  with  $\chi_{[S_\delta]} \partial_x \tilde{p}$ , which provides (5.23a). The convergence expressed in (5.23b) can be proven by similar computations.

The weak convergence expressed in (5.24a) and (5.24b) can be established analogously to (5.22d). Therefore, it remains to show that the fluxes  $\tilde{J}^x$  and  $\tilde{J}^y$  coincide with  $\tilde{u} \partial_x \tilde{p}$  and  $\tilde{u} \partial_y \tilde{p}$ , respectively. Reusing the ideas of the proof of (5.23a), we choose  $\tilde{\zeta} \in L^\infty(\tilde{\Omega}; C^\infty([0, T_{\max}]; C_{\text{per}}^\infty(\overline{\mathcal{O}})))$  with  $\text{supp } \tilde{\zeta}(\omega, t) \subset S_\delta(\omega, t)$  and compute

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{J}_h^y \tilde{\zeta} \, dx \, dy \, dt \right] &= \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \tilde{p}_h \right\} \tilde{\zeta} \, dx \, dy \, dt \right] \\ &= \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \tilde{p}_h \right\} \tilde{\zeta} \, dx \, dy \, dt \right] \\ &\quad + \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} (1 - \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}) \chi_{[S_\delta]} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \tilde{p}_h \right\} \tilde{\zeta} \, dx \, dy \, dt \right] \\ &= \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \tilde{p}_h \tilde{\zeta} \, dx \, dy \, dt \right] \\ &\quad - \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \chi_{[S_\delta]} (I - \mathcal{I}_h^y) \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \tilde{p}_h \right\} \tilde{\zeta} \, dx \, dy \, dt \right] \\ &\quad + \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} (1 - \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}) \chi_{[S_\delta]} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \tilde{p}_h \right\} \tilde{\zeta} \, dx \, dy \, dt \right] \\ &=: B_1 + B_2 + B_3 . \end{aligned} \quad (5.40)$$

The convergence

$$B_1 \rightarrow \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \chi_{[S_\delta]} \tilde{u} \partial_x \tilde{p} \tilde{\zeta} \, dx \, dy \, dt \right] = \tilde{\mathbb{E}} \left[ \int_0^{T_{\max}} \int_{\mathcal{O}} \tilde{u} \partial_x \tilde{p} \tilde{\zeta} \, dx \, dy \, dt \right] \quad (5.41)$$

follows directly from (5.22e) and (5.23a). Combining (A.1c) (cf. Lemma A.1) with standard inverse estimates (cf. Theorem 4.5.11 in [10]) and (5.30) provides the estimate

$$\begin{aligned} B_2 &\leq Ch \left\| \partial_y \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \|\partial_x \tilde{p}_h\|_{L^2(S_\delta)} \\ &\leq C\delta^{-1} \left( \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} - \tilde{u}_h \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} + h \|\partial_y \tilde{u}_h\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))} \right). \end{aligned} \quad (5.42)$$

Therefore,  $B_2$  vanishes for  $h \searrow 0$  due to (5.22e), and (5.22a). Applying Hölder's inequality and the bounds established in Proposition 5.3, we obtain

$$B_3 \leq C \left\| \left( \chi_{[S_\delta]} - \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]} \right) \tilde{\zeta} \right\|_{L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))}. \quad (5.43)$$

Similar to the arguments used in the proof of (5.23a), we may use Egorov's theorem to obtain the existence of a subset  $\mathfrak{E}_\iota \subset \tilde{\Omega} \times (0, T_{\max})$  with measure smaller than  $\iota$  such that  $\tilde{u}_h$  converges uniformly in  $C_{\text{per}}^{0,\gamma}(\overline{\mathcal{O}})$  in  $(\tilde{\Omega} \times (0, T_{\max})) \setminus \mathfrak{E}_\iota =: \mathfrak{E}_\iota^c$ . As we have  $\chi_{[S_\delta]} \subset \chi_{[S_{\delta/4}^{\mathcal{Q}_h}]}$  in  $\mathfrak{E}_\iota^c$  for  $h$  small enough and  $\text{supp } \tilde{\zeta} \subset S_\delta$ , we obtain  $B_3 \leq C\iota^{1/2}$  for all  $\iota > 0$ . As we also have  $\tilde{J}_h^x \rightharpoonup \tilde{J}^x$  in  $L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O})))$ , we obtain  $\tilde{J}^x = \tilde{u} \partial_x (-\Delta \tilde{u} + F'(\tilde{u}))$  on  $S_\delta$ . The identification of  $\tilde{J}^y$  on  $S_\delta$  follows by similar arguments.  $\square$

**5.3. Convergence of the stochastic integral.** We consider for arbitrary but fixed  $v \in C_{\text{per}}^\infty(\overline{\mathcal{O}})$  the operator  $\mathcal{M}_{h,v} : \Omega \times [0, T_{\max}] \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \mathcal{M}_{h,v}(t) &:= \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (u_h(t) - u_h(0))v \} \, dx \, dy \\ &\quad + \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(u_h)]_x^{-1}} J_h^x \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} \, dx \, dy \, d\tau \\ &\quad + \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \sqrt{[G''(u_h)]_y^{-1}} J_h^y \partial_y \mathcal{I}_h^{xy} \{ v \} \right\} \, dx \, dy \, d\tau \\ &= \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \, d\beta_{kl}^x \\ &\quad + \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \, d\beta_{kl}^y. \end{aligned} \quad (5.44)$$

**Remark 5.11.** *In contrast to [28], we define  $\mathcal{M}_{h,v}$  using functions  $v$  in  $C_{\text{per}}^\infty(\overline{\mathcal{O}})$  instead of  $H_{\text{per}}^2(\mathcal{O})$ . This seems to be necessary, as the limit process in Lemma 5.14 requires convergence and stability properties of the projection of  $v$  that are not provided by the  $L^2$ -projection. Hence, the requirements on the regularity of  $v$  are also higher.*

By the optional stopping theorem,  $\mathcal{M}_{h,v}$  is a real valued martingale, i.e. we have

$$\mathbb{E}[(\mathcal{M}_{h,v}(t) - \mathcal{M}_{h,v}(s))\Psi(r_s u_h, r_s \mathbf{W})] = 0 \quad (5.45)$$

for all  $0 \leq s \leq t \leq T_{\max}$  and for all  $[0, 1]$ -valued functions  $\Psi$  defined on  $\mathcal{X}_u$ . Here,  $r_s$  denotes the restriction of a function on  $[0, T_{\max}]$  onto  $[0, s]$ .

**Lemma 5.12.** *For the quadratic variation of  $\mathcal{M}_{h,v}$ , we have*

$$\begin{aligned} \langle\langle \mathcal{M}_{h,v} \rangle\rangle_t &= \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge T_h} \lambda_{kl}^x{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 ds \\ &\quad + \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge T_h} \lambda_{kl}^y{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 ds \quad (5.46) \\ &\leq C \|v\|_{H^2(\mathcal{O})}^2 \int_0^{t \wedge T_h} \|u_h(s)\|_{H^1(\mathcal{O})}^2 ds. \end{aligned}$$

*Proof.* We consider the mapping  $R(u_h, v) : \Omega \times [0, T_{\max}] \times (L^2(\mathcal{O}))^2 \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} (\omega, t, (z_x, z_y)) &\mapsto \chi_{T_h}(t, \omega) \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (u_h \sum_{k \in I_h^x} \sum_{l \in I_h^y} \langle \mathfrak{g}_{kl}, z_x \rangle_{L^2} \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \right\} \right\} dx dy \\ &\quad + \chi_{T_h}(t, \omega) \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^y \left\{ \partial_y (u_h \sum_{k \in I_h^x} \sum_{l \in I_h^y} \langle \mathfrak{g}_{kl}, z_y \rangle_{L^2} \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \right\} \right\} dx dy. \quad (5.47) \end{aligned}$$

We obtain for the Hilbert-Schmidt norm

$$\begin{aligned} \|R(u_h, v)(t, \omega)\|_{L_2(Q^{1/2}(L^2(\mathcal{O}))^2; \mathbb{R})}^2 &= \chi_{T_h}(t, \omega) \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 \\ &\quad + \chi_{T_h}(t, \omega) \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^y{}^2 \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 \\ &=: \chi_{T_h}(t, \omega) \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 A^2 + \chi_{T_h}(t, \omega) \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^y{}^2 B^2. \quad (5.48) \end{aligned}$$

To estimate the first term, we use (2.13a) and (2.13b) and compute

$$\begin{aligned} A^2 &\leq 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x u_h \mathcal{I}_h^x \{ \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 + 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ u_h \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 \\ &\leq C \|\tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2 \|\partial_x u_h\|_{L^2(\mathcal{O})}^2 \|\mathcal{I}_h^{xy} \{ v \}\|_{L^2(\mathcal{O})}^2 + \|\partial_x \tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})}^2 \|u_h\|_{L^2(\mathcal{O})}^2 \|\mathcal{I}_h^{xy} \{ v \}\|_{L^2(\mathcal{O})}^2. \quad (5.49) \end{aligned}$$

Using **(B3)**, the standard error estimates for  $\mathcal{I}_h^{xy}$ , and a similar estimate for  $B^2$ , we conclude

$$\|R(u_h, v)(t, \omega)\|_{L_2(Q^{1/2}(L^2(\mathcal{O}))^2; \mathbb{R})}^2 \leq C \chi_{T_h}(t, \omega) \|u_h\|_{H^1(\mathcal{O})}^2 \|v\|_{H^2(\mathcal{O})}^2. \quad (5.50)$$

Applying Lemma 2.4.3 in [61] concludes the proof.  $\square$

In the next lemma, we will study cross variation of  $\mathcal{M}_{h,v}$  with the processes

$$\beta_{kl}^\alpha(t) = \int_{\mathcal{O}} \int_0^t (\lambda_{kl}^\alpha)^{-1} \mathbf{g}_{kl} \mathbf{b}_\alpha \cdot d\mathbf{W} \, dx \, dy \quad (5.51)$$

for  $k, l \in \mathbb{Z}$  and  $\alpha \in \{x, y\}$ .

**Lemma 5.13.** *For  $k, l \in \mathbb{Z}$  and  $\alpha \in \{x, y\}$ , the cross variation  $\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^\alpha \rangle\rangle_t$  is given by*

$$\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^x \rangle\rangle_t = \begin{cases} \lambda_{kl}^x \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \, ds & \text{if } k \in I_h^x, l \in I_h^y, \\ 0 & \text{else,} \end{cases} \quad (5.52a)$$

$$\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^y \rangle\rangle_t = \begin{cases} \lambda_{kl}^y \int_0^{t \wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \, ds & \text{if } k \in I_h^x, l \in I_h^y, \\ 0 & \text{else.} \end{cases} \quad (5.52b)$$

*Proof.* The proof follows the lines of Lemma 5.12 in [28]. We will only prove (5.52a), as (5.52b) follows by similar computations.

To compute the cross variation with  $\beta_{kl}^x$ , we consider for given  $k, l$  the mappings  $S_\pm^x : \Omega \times [0, T_{\max}] \times Q_x^{1/2} L^2(\Omega) \rightarrow \mathbb{R}$  which are defined as

$$z \mapsto \chi_{T_h} \sum_{\bar{k} \in I_h^x} \sum_{\bar{l} \in I_h^y} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \langle \mathbf{g}_{\bar{k}\bar{l}}, z \rangle_{L^2}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \pm (\lambda_{kl}^x)^{-1} \int_{\mathcal{O}} \mathbf{g}_{kl} z \, dx \, dy \right). \quad (5.53)$$

Computing the Hilbert-Schmidt norm of  $S_\pm^x$ , we obtain

$$\begin{aligned} \|S_\pm^x(u_h, v)\|_{L_2(Q_x^{1/2} L^2(\mathcal{O}); \mathbb{R})} &= \chi_{T_h} \sum_{\bar{k} \in I_h^x} \sum_{\bar{l} \in I_h^y} \lambda_{\bar{k}\bar{l}}^x \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,\bar{k}\bar{l}}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \right)^2 \\ &\pm \chi_{T_h} 2 \sum_{\bar{k} \in I_h^x} \sum_{\bar{l} \in I_h^y} \lambda_{\bar{k}\bar{l}}^x \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,\bar{k}\bar{l}}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \frac{\lambda_{\bar{k}\bar{l}}^x}{\lambda_{kl}^x} \int_{\mathcal{O}} \mathbf{g}_{kl} \mathbf{g}_{\bar{k}\bar{l}} \, dx \, dy \\ &+ \chi_{T_h} \sum_{\bar{k}, \bar{l} \in \mathbb{Z}} \left( \frac{\lambda_{\bar{k}\bar{l}}^x}{\lambda_{kl}^x} \int_{\mathcal{O}} \mathbf{g}_{kl} \mathbf{g}_{\bar{k}\bar{l}} \, dx \, dy \right)^2. \end{aligned} \quad (5.54)$$

Using  $\langle\langle \mathcal{M}_{h,v}, \beta_{kl}^x \rangle\rangle_t = \frac{1}{4} (\langle\langle S_+^x(u_h, v) \rangle\rangle_t - \langle\langle S_-^x(u_h, v) \rangle\rangle_t)$  and recalling the identity  $\int_{\mathcal{O}} \mathbf{g}_{kl} \mathbf{g}_{\bar{k}\bar{l}} \, dx \, dy = \delta_{\bar{k}\bar{l}} \delta_{\bar{l}\bar{l}}$ , we deduce (5.52a).  $\square$

In addition to  $\mathcal{M}_{h,v}$ , the processes

$$\begin{aligned} \mathcal{M}_{h,v}^2 &- \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{(\cdot) \wedge T_h} \lambda_{kl}^x 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (u_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \right)^2 \, ds \\ &- \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{(\cdot) \wedge T_h} \lambda_{kl}^y 2 \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (u_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} \, dx \, dy \right)^2 \, ds \end{aligned} \quad (5.55)$$

and

$$\mathcal{M}_{h,v}\beta_{kl}^x - \lambda_{kl}^x \int_0^{(\cdot)\wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy ds \quad \begin{array}{l} \text{if } k \in I_h^x, l \in I_h^y, \\ \text{else,} \end{array} \quad (5.56)$$

$$\mathcal{M}_{h,v}\beta_{kl}^y - \lambda_{kl}^y \int_0^{(\cdot)\wedge T_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^y \left\{ \partial_y (u_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy ds \quad \begin{array}{l} \text{if } k \in I_h^x, l \in I_h^y, \\ \text{else} \end{array} \quad (5.57)$$

are also a martingales.

By equality of laws, we deduce that the following processes are also  $(\tilde{\mathcal{F}}_{h,t})$ -martingales:

$$\begin{aligned} \tilde{\mathcal{M}}_{h,v}(t) &:= \int_{\mathcal{O}} \mathcal{I}_h^{xy} \left\{ (\tilde{u}_h(t) - \tilde{u}_h(0)) \mathcal{I}_h^{xy} \{v\} \right\} dx dy \\ &+ \int_0^{t\wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{v\} \right\} dx dy d\tau \\ &+ \int_0^{t\wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \tilde{J}_h^y \partial_y \mathcal{I}_h^{xy} \{v\} \right\} dx dy d\tau, \end{aligned} \quad (5.58a)$$

$$\begin{aligned} \tilde{\mathcal{M}}_{h,v}^2(t) &- \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 \int_0^{t\wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy \right)^2 ds \\ &- \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^y{}^2 \int_0^{t\wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^y \left\{ \partial_y (\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy \right)^2 ds, \end{aligned} \quad (5.58b)$$

$$\tilde{\mathcal{M}}_{h,v}\tilde{\beta}_{h,kl}^x - \lambda_{kl}^x \int_0^{(\cdot)\wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \mathcal{I}_{h,\text{loc}}^x \left\{ \partial_x (\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy ds \quad \begin{array}{l} \text{if } k \in I_h^x, l \in I_h^y, \\ \text{else,} \end{array} \quad (5.58c)$$

$$\tilde{\mathcal{M}}_{h,v}\tilde{\beta}_{h,kl}^y - \lambda_{kl}^y \int_0^{(\cdot)\wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \mathcal{I}_{h,\text{loc}}^y \left\{ \partial_y (\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \right\} \right\} dx dy ds \quad \begin{array}{l} \text{if } k \in I_h^x, l \in I_h^y, \\ \text{else.} \end{array} \quad (5.58d)$$

Here, we used

$$\tilde{\beta}_{h,kl}^\alpha(t) = \int_{\mathcal{O}} \int_0^t (\lambda_{kl}^\alpha)^{-1} \mathfrak{g}_{kl} \mathbf{b}_\alpha \cdot d\tilde{\mathbf{W}}_h dx dy \quad (5.59)$$

for  $k, l \in \mathbb{Z}$  and  $\alpha \in \{x, y\}$ . Furthermore, the quadratic variation of  $\tilde{\mathcal{M}}_{h,v}$  and the cross variations with  $\tilde{\beta}_{h,kl}^\alpha$  are given as

$$\begin{aligned} \langle\langle \tilde{\mathcal{M}}_{h,v} \rangle\rangle_t &= \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge \tilde{T}_h} \lambda_{kl}^{x,2} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\tilde{u}_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 ds \\ &\quad + \sum_{k \in I_h^x} \sum_{l \in I_h^y} \int_0^{t \wedge \tilde{T}_h} \lambda_{kl}^{y,2} \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\tilde{u}_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy \right)^2 ds \end{aligned} \quad (5.60)$$

$$\langle\langle \tilde{\mathcal{M}}_{h,v}, \tilde{\beta}_{h,kl}^x \rangle\rangle_t = \begin{cases} \lambda_{kl}^x \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x (\tilde{u}_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy ds & \text{if } k \in I_h^x, l \in I_h^y, \\ 0 & \text{else,} \end{cases} \quad (5.61)$$

$$\langle\langle \tilde{\mathcal{M}}_{h,v}, \tilde{\beta}_{h,kl}^y \rangle\rangle_t = \begin{cases} \lambda_{kl}^y \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y (\tilde{u}_h \tilde{\mathbf{g}}_{h,kl}) \mathcal{I}_h^{xy} \{ v \} \} \} dx dy ds & \text{if } k \in I_h^x, l \in I_h^y, \\ 0 & \text{else.} \end{cases} \quad (5.62)$$

**Lemma 5.14.** *Let the Assumptions (S), (I), (P), (B), (R), and (B3\*) hold true. Then, for all  $[0, 1]$ -valued continuous functions  $\Psi$  defined on  $\mathcal{X}_u \times \mathcal{X}_W$ , we have*

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left( \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(s)) v dx dy + \int_s^t \int_{\mathcal{O}} \tilde{u} J^x \partial_x v dx dy d\tau \right. \right. \\ \left. \left. + \int_s^t \int_{\mathcal{O}} \tilde{u} J^y \partial_y v dx dy d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right] = 0 \end{aligned} \quad (5.63)$$

for all  $0 \leq s \leq t \leq T_{\max}$ .

*Proof.* To establish the claim, we pass to the limit in the identity

$$\tilde{\mathbb{E}} \left[ \left( \tilde{\mathcal{M}}_{h,v}(t) - \tilde{\mathcal{M}}_{h,v}(s) \right) \Psi(r_s \tilde{u}_h, r_s \tilde{W}_h) \right] = 0. \quad (5.64)$$

Due to Lemma 5.5, we have  $\chi_{\tilde{T}_h} = 1$  on  $[s, t]$  for  $h$  small enough depending on  $\tilde{\omega} \in \tilde{\Omega}$ . We start with the decomposition

$$\begin{aligned} &\int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ (\tilde{u}_h(t) - \tilde{u}_h(s)) v \} dx dy \\ &= \int_{\mathcal{O}} (\tilde{u}_h(t) - \tilde{u}_h(s)) \mathcal{I}_h^{xy} \{ v \} dx dy - \int_{\mathcal{O}} (I - \mathcal{I}_h^{xy}) \{ (\tilde{u}_h(t) - \tilde{u}_h(s)) \mathcal{I}_h^{xy} \{ v \} \} dx dy. \end{aligned} \quad (5.65)$$

Due to the strong convergence of  $\tilde{u}_h$  in  $C([0, T_{\max}]; L^q(\mathcal{O}))$   $\tilde{\mathbb{P}}$ -almost surely and the convergence properties of  $\mathcal{I}_h^{xy}$  (cf. Theorem 4.4.20 in [10]), the first term on the right-hand side converges  $\tilde{\mathbb{P}}$ -almost surely towards  $\int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(s)) v dx dy$ , while the second term

vanishes due to Lemma A.1, as the following computation shows.

$$\begin{aligned} \left| \int_{\mathcal{O}} (I - \mathcal{I}_h^{xy}) \{ (\tilde{u}_h(t) - \tilde{u}_h(s)) \mathcal{I}_h^{xy} \{ v \} \} dx dy \right| &\leq Ch \|\tilde{u}_h(t) - \tilde{u}_h(s)\|_{L^1(\mathcal{O})} \|\nabla \mathcal{I}_h^{xy} \{ v \}\|_{L^\infty(\mathcal{O})} \\ &\leq Ch \int_{\mathcal{O}} \tilde{u}_h^0 dx dy \|\nabla v\|_{L^\infty(\mathcal{O})} \leq Ch. \end{aligned} \quad (5.66)$$

Here, we used a standard inverse estimate (cf. Theorem 4.5.11 in [10]) and the non-negativity of  $\tilde{u}_h$ . In order to deal with the remaining terms, we use the decomposition

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} dx dy dt \\ &= \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{ v \} dx dy dt \\ &\quad - \int_{t_1}^{t_2} \int_{\mathcal{O}} (I - \mathcal{I}_h^y) \left\{ \tilde{J}_h^x \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} \right\} dx dy dt \\ &\quad - \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{J}_h^x (I - \mathcal{I}_h^y) \left\{ \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} dx dy dt \\ &= A + B + C. \end{aligned} \quad (5.67)$$

Applying (A.1c) from Lemma A.1 pointwise in  $x$  together with an inverse estimate (cf. Theorem 4.5.11 [10]) and applying Hölder's inequality, we obtain

$$|B| \leq Ch \int_{t_1}^{t_2} \left\| \tilde{J}_h^x \right\|_{L^2(\mathcal{O})} \left\| \partial_y \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} \right\|_{L^2(\mathcal{O})} dt. \quad (5.68)$$

Applying (A.1d) from Lemma A.1 and standard inverse estimates (cf. Theorem 4.5.11 in [10]), we deduce

$$\begin{aligned} &\left\| \partial_y \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} \right\|_{L^2(\mathcal{O})} \\ &\leq \left\| \partial_y \left( \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \mathcal{I}_h^{xy} \{ v \} \right) \right\|_{L^2(\mathcal{O})} \\ &\quad + \left\| \partial_y (I - \mathcal{I}_h^y) \left\{ \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \mathcal{I}_h^{xy} \{ v \} \right\} \right\|_{L^2(\mathcal{O})} \\ &\leq \left\| \partial_y \left( \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right) \partial_x \mathcal{I}_h^{xy} \{ v \} \right\|_{L^2(\mathcal{O})} + \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_y \partial_x \mathcal{I}_h^{xy} \{ v \} \right\|_{L^2(\mathcal{O})} \\ &\quad + C \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right\|_{L^2(\mathcal{O})} \|\partial_y \partial_x \mathcal{I}_h^{xy} \{ v \}\|_{L^\infty(\mathcal{O})}. \end{aligned} \quad (5.69)$$

As  $v$  was chosen to be sufficiently regular to control  $\|\partial_y \partial_x \mathcal{I}_h^{xy} \{ v \}\|_{L^\infty(\mathcal{O})}$ , the only problematic term is the first one on the right-hand side. In view of (5.22e), we use the regularity of  $v$  and apply an inverse estimate (cf. Theorem 4.5.11 in [10]) to obtain

$$\begin{aligned}
& \left\| \partial_y \left( \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right) \partial_x \mathcal{I}_h^{xy} \{v\} \right\|_{L^2(\mathcal{O})} \\
& \leq C \left\| \partial_y \left( \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} - \tilde{u}_h \right) \right\|_{L^2(\mathcal{O})} + C \|\partial_y \tilde{u}_h\|_{L^2(\mathcal{O})} \\
& \leq Ch^{-1} \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} - \tilde{u}_h \right\|_{L^2(\mathcal{O})} + C \|\partial_y \tilde{u}_h\|_{L^2(\mathcal{O})}. \quad (5.70)
\end{aligned}$$

In conclusion, by Hölder's inequality, we have

$$\begin{aligned}
|B| & \leq Ch \left\| \tilde{J}_h^x \right\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \left( \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} + \|\partial_y \tilde{u}_h\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \right) \\
& + C \left\| \tilde{J}_h^x \right\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} - \tilde{u}_h \right\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))}. \quad (5.71)
\end{aligned}$$

Therefore, in view of Lemma 5.10,  $B$  vanishes in  $L^q(\tilde{\Omega})$  for any  $q < \infty$ . In the same spirit, we use Hölder's inequality and (A.1c) in Lemma A.1 to prove that  $C$  vanishes in  $L^q(\tilde{\Omega})$ .

$$\begin{aligned}
|C| & \leq \int_{t_1}^{t_2} \left\| \tilde{J}_h^x \right\|_{L^2(\mathcal{O})} \left\| (I - \mathcal{I}_h^y) \left\{ \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \partial_x \mathcal{I}_h^{xy} \{v\} \right\} \right\|_{L^2(\mathcal{O})} dt \\
& \leq Ch \int_{t_1}^{t_2} \left\| \tilde{J}_h^x \right\|_{L^2(\mathcal{O})} \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right\|_{L^2(\mathcal{O})} \|\partial_y \partial_x \mathcal{I}_h^{xy} \{v\}\|_{L^\infty(\mathcal{O})} dt \\
& \leq Ch. \quad (5.72)
\end{aligned}$$

Therefore, it remains to analyze the convergence properties of  $A$ . As  $\tilde{J}_h^x$  converges weakly towards  $\tilde{J}^x$   $\tilde{\mathbb{P}}$ -almost surely in  $L^2(0, T_{\max}; L^2(\mathcal{O}))$ , we may use the strong convergence of  $\mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\}$  in  $L^\infty(0, T_{\max}; L^q(\mathcal{O}))$  with  $q < \infty$  (cf. (5.22e) in Lemma 5.10) and  $\partial_x \mathcal{I}_h^{xy} \{v\}$  in  $L^\infty(\mathcal{O})$  to conclude that

$$A \rightarrow \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{u} \tilde{J}^x \partial_x v \, dx \, dy \, dt \quad \tilde{\mathbb{P}}\text{-almost surely.} \quad (5.73)$$

Analogous arguments provide the  $\tilde{\mathbb{P}}$ -almost sure convergence

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{v\} \, dx \, dy \, dt \rightarrow \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{u} \tilde{J}^y \partial_y v \, dx \, dy \, dt. \quad (5.74)$$

As the  $\Psi$ -term is continuous, uniformly bounded, and converges  $\tilde{\mathbb{P}}$ -almost surely, it remains to control higher moments to conclude the proof by applying Vitali's convergence theorem.

As  $\tilde{u}_h$  is uniformly bounded in  $L^q(\tilde{\Omega}; C([0, T_{\max}]; L^q(\mathcal{O})))$  for arbitrary  $q < \infty$  and the error terms  $B$  and  $C$ , which were introduced by the nodal interpolation operators, vanish also in  $L^q(\tilde{\Omega})$ , it remains to establish the integrability of appropriate higher moments of

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{v\} \, dx \, dy \, dt \quad (5.75a)$$

$$\text{and} \quad \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \right\} \tilde{J}_h^y \partial_y \mathcal{I}_h^{xy} \{v\} \, dx \, dy \, dt. \quad (5.75b)$$

The desired integrability of higher moments of the first integral follows from the estimate

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{v\} \, dx \, dy \, dt \right| \\ & \leq C \left\| \tilde{J}_h^x \right\|_{L^2(0, T_{\max}; L^2(\mathcal{O}))} \left\| \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \right\} \right\|_{L^\infty(0, T_{\max}; L^q(\mathcal{O}))} \left\| \partial_x \mathcal{I}_h^{xy} \{v\} \right\|_{L^\infty(\mathcal{O})}. \end{aligned} \quad (5.76)$$

Together with the bounds from Proposition 5.3 and Lemma 5.10, we obtain the uniform integrability of a  $q$ -moment for  $q > 1$ . A similar argument provides the uniform integrability of the second term in (5.75b), which concludes the proof.  $\square$

**Lemma 5.15.** *Let the Assumptions (S), (I), (P), (B), (R), (B3\*), and (B4) hold true. Then for all  $[0, 1]$ -valued continuous functions  $\Psi$  defined on  $\mathcal{X}_u \times \mathcal{X}_W$ , we have*

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left( \tilde{\mathcal{M}}_v^2(t) - \tilde{\mathcal{M}}_v^2(s) - \int_s^t \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^x \left( \int_{\mathcal{O}} \tilde{u} \mathbf{g}_{kl} \partial_x v \, dx \, dy \right)^2 \, d\tau \right. \right. \\ \left. \left. - \int_s^t \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^y \left( \int_{\mathcal{O}} \tilde{u} \mathbf{g}_{kl} \partial_y v \, dx \, dy \right)^2 \, d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right] = 0 \end{aligned} \quad (5.77)$$

for all  $0 \leq s \leq t \leq T_{\max}$ , where

$$\tilde{\mathcal{M}}_v(t) := \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0))v \, dx \, dy + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^x \partial_x v \, dx \, dy \, d\tau + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^y \partial_y v \, dx \, dy \, d\tau. \quad (5.78)$$

*Proof.* We will prove this by passing to the limit in the martingale (5.58b). Recalling the arguments from the proof of Lemma 5.14, we obtain that

$$\begin{aligned} \tilde{\mathcal{M}}_{h,v}^2(t) & := \left( \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{(\tilde{u}_h(t) - \tilde{u}_h(0))v\} \, dx \, dy \right. \\ & \quad + \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y \left\{ \sqrt{[G''(\tilde{u}_h)]_x^{-1}} \tilde{J}_h^x \partial_x \mathcal{I}_h^{xy} \{v\} \right\} \, dx \, dy \, d\tau \\ & \quad \left. + \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x \left\{ \sqrt{[G''(\tilde{u}_h)]_y^{-1}} \tilde{J}_h^y \partial_y \mathcal{I}_h^{xy} \{v\} \right\} \, dx \, dy \, d\tau \right)^2 \end{aligned} \quad (5.79)$$

converges along a subsequence  $\tilde{\mathbb{P}}$ -almost surely towards

$$\begin{aligned} \tilde{\mathcal{M}}_v^2(t) & = \left( \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0))v \, dx \, dy \right. \\ & \quad \left. + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^x \partial_x v \, dx \, dy \, d\tau + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^y \partial_y v \, dx \, dy \, d\tau \right)^2. \end{aligned} \quad (5.80)$$

In order to deduce the convergence of the corresponding expected values, we need to establish higher regularity of  $\tilde{\mathcal{M}}_{h,v}$ . Starting from the representation

$$\begin{aligned} \tilde{\mathcal{M}}_{h,v}(t) = & - \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \} \} dx dy d\tilde{\beta}_{h,kl}^x \\ & - \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^y \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \} \} dx dy d\tilde{\beta}_{h,kl}^y \end{aligned} \quad (5.81)$$

and combining the martingale moment inequality

$$\tilde{\mathbb{E}} \left[ |\tilde{\mathcal{M}}_{h,v}(t)|^{2q} \right] \leq C_q \tilde{\mathbb{E}} \left[ \left\langle \left\langle \tilde{\mathcal{M}}_{h,v} \right\rangle \right\rangle_t^q \right] \quad \text{for any } q > 0 \quad (5.82)$$

(see e.g. Proposition 3.26 in Chapter 3 of [53]) with Lemma 5.12 formulated for  $\tilde{\mathcal{M}}_{h,v}$ , we obtain

$$\begin{aligned} \tilde{\mathbb{E}} \left[ |\tilde{\mathcal{M}}_{h,v}(t)|^{2q} \right] & \leq C \tilde{\mathbb{E}} \left[ \left\langle \left\langle \tilde{\mathcal{M}}_{h,v} \right\rangle \right\rangle_t^q \right] \leq C \|v\|_{H^2(\mathcal{O})}^{2q} \tilde{\mathbb{E}} \left[ \left( \int_0^{t \wedge \tilde{T}_h} \|\tilde{u}_h(s)\|_{H^1(\mathcal{O})}^2 ds \right)^q \right] \\ & \leq C \|v\|_{H^2(\mathcal{O})}^{2q} T_{\max}^q \tilde{\mathbb{E}} \left[ \sup_{s \in [0, T_{\max}]} \mathcal{E}_h(\tilde{u}_h)^q + C \right]. \end{aligned} \quad (5.83)$$

It remains to pass to the limit in the remaining integrals in (5.58b), i.e. we have to analyze the convergence behavior of  $\left\langle \left\langle \tilde{\mathcal{M}}_{h,v} \right\rangle \right\rangle_t$  for  $h \searrow 0$  (cf. (5.60)). The first step is to show that we may neglect the interpolation operators when passing to the limit. Applying (2.13a) and (2.13b), we may rewrite the first component of  $\tilde{\mathcal{M}}_{h,v}$  using

$$\begin{aligned} & \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \} \} dx dy \\ & = \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{u}_h \mathcal{I}_h^x \{ \tilde{\mathfrak{g}}_{h,kl} v \} \} dx dy + \int_{\mathcal{O}} \mathcal{I}_h^y \{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^x \{ \tilde{u}_h v \} \} dx dy \\ & = \int_{\mathcal{O}} \partial_x \tilde{u}_h \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^{xy} \{v\} dx dy + \int_{\mathcal{O}} \partial_x \tilde{\mathfrak{g}}_{h,kl} \tilde{u}_h \mathcal{I}_h^{xy} \{v\} dx dy \\ & - \int_{\mathcal{O}} (I - \mathcal{I}_h^y) \{ \partial_x \tilde{u}_h \mathcal{I}_h^{xy} \{ \tilde{\mathfrak{g}}_{h,kl} v \} \} dx dy - \int_{\mathcal{O}} \partial_x \tilde{u}_h (I - \mathcal{I}_h^{xy}) \{ \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^{xy} \{v\} \} dx dy \\ & - \int_{\mathcal{O}} (I - \mathcal{I}_h^y) \{ \partial_x \tilde{\mathfrak{g}}_{h,kl} \mathcal{I}_h^{xy} \{ \tilde{u}_h v \} \} dx dy - \int_{\mathcal{O}} \partial_x \tilde{\mathfrak{g}}_{h,kl} (I - \mathcal{I}_h^{xy}) \{ \tilde{u}_h \mathcal{I}_h^{xy} \{v\} \} dx dy \\ & =: \int_{\mathcal{O}} \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} dx dy + A_{kl} + B_{kl} + C_{kl} + D_{kl}, \end{aligned} \quad (5.84)$$

where we used that  $\tilde{\mathfrak{g}}_{h,kl} \in U_h$ . Combining (5.84) with the binomial theorem and (5.60), we get

$$\left| \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t \wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ \mathcal{I}_{h,\text{loc}}^x \{ \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \} \} dx dy \right)^2 d\tau \right.$$

$$\begin{aligned}
& - \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 \int_0^{t \wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \, dx \, dy \right)^2 \, d\tau \Big| \\
& \leq C \sqrt{\langle \langle \tilde{\mathcal{M}}_{h,v} \rangle \rangle_t} \left( \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 \int_0^{t \wedge \tilde{T}_h} A_{kl}^2 + B_{kl}^2 + C_{kl}^2 + D_{kl}^2 \, d\tau \right)^{1/2} \\
& \quad + C \sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x{}^2 \int_0^{t \wedge \tilde{T}_h} A_{kl}^2 + B_{kl}^2 + C_{kl}^2 + D_{kl}^2 \, d\tau =: (\#). \quad (5.85)
\end{aligned}$$

Using Lemma A.1 and standard inverse estimates (cf. Theorem 4.5.11 in [10]), we obtain the estimates

$$A_{kl} \leq Ch \|\partial_x \tilde{u}_h\|_{L^2(\mathcal{O})} \|\partial_y \mathcal{I}_h^{xy} \{\tilde{\mathfrak{g}}_{h,kl} v\}\|_{L^2(\mathcal{O})} \leq Ch \|\partial_x \tilde{u}_h\|_{L^2(\mathcal{O})} \|\partial_y(\tilde{\mathfrak{g}}_{h,kl} v)\|_{L^\infty(\mathcal{O})}, \quad (5.86)$$

$$B_{kl} \leq Ch \|\partial_x \tilde{u}_h\|_{L^2(\mathcal{O})} \|\tilde{\mathfrak{g}}_{h,kl}\|_{L^4(\mathcal{O})} \|\nabla \mathcal{I}_h^{xy} \{v\}\|_{L^4(\mathcal{O})}, \quad (5.87)$$

$$\begin{aligned}
C_{kl} & \leq Ch \|\partial_x \partial_y \tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})} \|\mathcal{I}_h^{xy} \{\tilde{u}_h v\}\|_{L^1(\mathcal{O})} \\
& \leq Ch \|\partial_x \partial_y \tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})} \|\tilde{u}_h\|_{L^2(\mathcal{O})} \|\mathcal{I}_h^{xy} \{v\}\|_{L^2(\mathcal{O})}, \quad (5.88)
\end{aligned}$$

$$D_{kl} \leq Ch \|\partial_x \tilde{\mathfrak{g}}_{h,kl}\|_{L^\infty(\mathcal{O})} \|\tilde{u}_h\|_{L^2(\mathcal{O})} \|\nabla \mathcal{I}_h^{xy} \{v\}\|_{L^2(\mathcal{O})}. \quad (5.89)$$

Recalling Assumption **(B3)**, (2.6), and the regularity of  $v$ , we obtain

$$(\#) \leq Ch^{1/2} \sqrt{\langle \langle \tilde{\mathcal{M}}_{h,v} \rangle \rangle_t} \left( \int_0^{t \wedge \tilde{T}_h} \|\tilde{u}_h\|_{H^1(\mathcal{O})}^2 \, d\tau \right)^{1/2} + Ch \int_0^{t \wedge \tilde{T}_h} \|\tilde{u}_h\|_{H^1(\mathcal{O})}^2 \, d\tau. \quad (5.90)$$

Therefore, the  $\bar{p}$ -th moment of the left-hand side of (5.85) vanishes, which in particular provides convergence  $\tilde{\mathbb{P}}$ -almost surely after restricting ourselves to appropriate subsequences. It remains to discuss the convergence properties of

$$\sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t \wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \partial_x(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \, dx \, dy \right)^2 \, d\tau. \quad (5.91)$$

After integrating by parts, we may use the standard error estimates for the nodal interpolation operator (cf. Theorem 4.4.20 in [10]) and Assumption **(B3)** to obtain the strong convergence of  $\lambda_{kl}^x \tilde{\mathfrak{g}}_{h,kl}$  towards  $\lambda_{kl}^x \mathfrak{g}_{kl}$  in  $L^\infty(\mathcal{O})$  and the strong convergence of  $\partial_x \mathcal{I}_h^{xy} \{v\}$  towards  $\partial_x v$  in  $L^\infty(\mathcal{O})$ . Together with (5.22a), this provides the  $\tilde{\mathbb{P}}$ -almost sure convergence. Similar considerations provide the convergence of

$$\sum_{k \in I_h^x} \sum_{l \in I_h^y} \lambda_{kl}^x \int_0^{t \wedge \tilde{T}_h} \left( \int_{\mathcal{O}} \mathcal{I}_h^x \{ \mathcal{I}_{h,\text{loc}}^y \{ \partial_y(\tilde{u}_h \tilde{\mathfrak{g}}_{h,kl}) \mathcal{I}_h^{xy} \{v\} \} \} \, dx \, dy \right)^2 \, d\tau$$

$\tilde{\mathbb{P}}$ -almost surely. As we already established the higher integrability of  $\langle \langle \tilde{\mathcal{M}}_{h,v} \rangle \rangle_t$  in (5.83), we may conclude by applying Vitali's theorem.  $\square$

In the same spirit, we get the following result.

**Lemma 5.16.** *Let the Assumptions (S), (I), (P), (B), (R), (B3\*), and (B4) hold true. Then for all  $[0, 1]$ -valued continuous functions  $\Psi$  defined on  $\mathcal{X}_u \times \mathcal{X}_W$ , we have*

$$\tilde{\mathbb{E}} \left[ \left( \tilde{\mathcal{M}}_v(t) \tilde{\beta}_{kl}^\alpha(t) - \tilde{\mathcal{M}}_v(s) \tilde{\beta}_{kl}^\alpha(s) - \lambda_{kl}^\alpha \int_s^t \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right] = 0 \quad (5.92)$$

for all  $k, l \in \mathbb{Z}$ ,  $\alpha \in \{x, y\}$ , and all  $s \leq t \in [0, T_{max}]$ .

**Lemma 5.17.** *Let the Assumptions (S), (I), (P), (B), (R), (B3\*), and (B4) hold true. Then, we have*

$$\tilde{\mathcal{M}}_v(t) = \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^x \int_0^t \int_{\mathcal{O}} \partial_x(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^x + \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^y \int_0^t \int_{\mathcal{O}} \partial_y(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^y. \quad (5.93)$$

*Proof.* As a martingale with vanishing quadratic variation is almost surely constant, it is sufficient to show that

$$\begin{aligned} 0 &= \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot) \right\rangle \right\rangle_T + \left\langle \left\langle \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \right\rangle \right\rangle_T \\ &\quad - 2 \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot), \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \right\rangle \right\rangle_T. \end{aligned} \quad (5.94)$$

To compute the last term on the right-hand side, we use the cross variation formula, which can be found e.g. in Lemma 2.16 in Section 3.2 of [53], to obtain

$$\begin{aligned} &\left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot), \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \int_0^{(\cdot)} \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \right\rangle \right\rangle_T \\ &= \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \int_0^T \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_{kl}^\alpha(\cdot) \right\rangle \right\rangle_s. \end{aligned} \quad (5.95)$$

Following the arguments in [28], it is possible to show that the process  $s \mapsto \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_{kl}^\alpha(\cdot) \right\rangle \right\rangle_s$  is absolutely continuous  $\tilde{\mathbb{P}}$ -almost surely and consequently

$$d \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_{kl}^\alpha(\cdot) \right\rangle \right\rangle_s = \lambda_{kl}^\alpha \int_{\mathcal{O}} \partial_\alpha(\tilde{u}(s) \mathbf{g}_{kl}) v \, dx \, dy \, ds. \quad (5.96)$$

Using the identities

$$\begin{aligned} \left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot) \right\rangle \right\rangle_T &= \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^\alpha \int_0^T \left( \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \right)^2 ds \\ &= \left\langle \left\langle \sum_{\alpha \in \{x, y\}} \sum_{k, l \in \mathbb{Z}} \lambda_{kl}^\alpha \int_0^{(\cdot)} \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \right\rangle \right\rangle_T, \end{aligned} \quad (5.97)$$

we conclude

$$\left\langle \left\langle \tilde{\mathcal{M}}_v(\cdot) - \sum_{\alpha \in \{x,y\}} \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^\alpha \int_0^{(\cdot)} \int_{\mathcal{O}} \partial_\alpha(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^\alpha \right\rangle \right\rangle_T \equiv 0. \quad (5.98)$$

□

Having established the previous results, we are now in the position to prove Theorem 3.5.

*Proof of Theorem 3.5.* From Proposition 5.3, Corollary 5.7, and Lemma 5.8, we infer the existence of a stochastic basis  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ , of a Wiener process

$$\tilde{\mathbf{W}}(t) = \sum_{\alpha \in \{x,y\}} \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^\alpha \mathbf{g}_{kl} \tilde{\beta}_{kl}^\alpha \mathbf{b}_\alpha, \quad (5.99)$$

and of random variables

$$\begin{aligned} \tilde{u} \in L^q(\tilde{\mathcal{O}}; L^\infty(0, T_{\max}; H_{\text{per}}^1(\mathcal{O}))) \cap L^2(\tilde{\Omega}; L^2(0, T_{\max}; H_{\text{per}}^2(\mathcal{O}))) \\ \cap L^\sigma(\tilde{\Omega}; C^{1/4}([0, T_{\max}]; (H_{\text{per}}^1(\mathcal{O}))')), \end{aligned} \quad (5.100a)$$

$$\tilde{J}^x \in L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))), \quad (5.100b)$$

$$\tilde{J}^y \in L^2(\tilde{\Omega}; L^2(0, T_{\max}; L^2(\mathcal{O}))) \quad (5.100c)$$

with  $q < \infty$  and  $\sigma < 8/5$ . As shown in Lemma 5.10, these random variables satisfy

$$\tilde{J}^x = \tilde{u} \partial_x(-\Delta \tilde{u} + F'(\tilde{u})) \quad \tilde{\mathbb{P}}\text{-almost surely in } [\tilde{u} > 0], \quad (5.101a)$$

$$\tilde{J}^y = \tilde{u} \partial_y(-\Delta \tilde{u} + F'(\tilde{u})) \quad \tilde{\mathbb{P}}\text{-almost surely in } [\tilde{u} > 0]. \quad (5.101b)$$

Furthermore, we have  $\Lambda = \tilde{\mathbb{P}} \circ (\tilde{u}^0)^{-1}$  by construction. Lemma 5.14 implies that

$$\tilde{\mathcal{M}}_v(t) = \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}^0) v \, dx \, dy + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^x \partial_x v \, dx \, dy \, ds + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^y \partial_y v \, dx \, dy \, ds \quad (5.102)$$

is an  $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ -martingale, and by Lemma 5.17, we obtain

$$\begin{aligned} & \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}^0) v \, dx \, dy + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^x \partial_x v \, dx \, dy \, ds + \int_0^t \int_{\mathcal{O}} \tilde{u} \tilde{J}^y \partial_y v \, dx \, dy \, ds \\ &= \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^x \int_0^t \int_{\mathcal{O}} \partial_x(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^x + \sum_{k,l \in \mathbb{Z}} \lambda_{kl}^y \int_0^t \int_{\mathcal{O}} \partial_y(\tilde{u} \mathbf{g}_{kl}) v \, dx \, dy \, d\tilde{\beta}_{kl}^y. \end{aligned} \quad (5.103)$$

It remains to establish the energy estimate (3.9). Starting from Proposition 4.2, using Fatou's lemma and the definition of the fluxes (5.1), we find

$$\begin{aligned} & \tilde{\mathbb{E}} \left[ \liminf_{h \searrow 0} \left( \sup_{t \in [0, T_{\max}]} \left( \frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x \tilde{u}_h|^2 \} + \mathcal{I}_h^x \{ |\partial_y \tilde{u}_h|^2 \} \, dx \, dy + \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(\tilde{u}_h) \} \, dx \, dy \right)^{\bar{p}} \right) \right] \\ & \quad + \tilde{\mathbb{E}} \left[ \liminf_{h \searrow 0} \int_0^{T_{\max}} \int_{\mathcal{O}} |\tilde{J}_h^x|^2 + |\tilde{J}_h^y|^2 \, dx \, dy \, dt \right] \end{aligned}$$

$$\leq \liminf_{h \searrow 0} \tilde{\mathbb{E}} \left[ \sup_{t \in [0, T_{\max}]} \mathcal{E}_h(\tilde{u}_h)^{\bar{p}} + \int_0^{T_{\max}} \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\tilde{J}_h^x|^2 \} + \mathcal{I}_h^x \{ |\tilde{J}_h^y|^2 \} dx dy dt \right] \leq C(\bar{p}, u^0, T_{\max}), \quad (5.104)$$

where we used  $\frac{1}{2}h^\varepsilon \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ |\Delta_h \tilde{u}_h|^2 \} dx dy \geq 0$ . By the norm equivalence (2.6), we obtain

$$\tilde{\mathbb{E}} \left[ \liminf_{h \searrow 0} \left( \sup_{t \in [0, T_{\max}]} \left( \frac{1}{2} \int_{\mathcal{O}} |\nabla \tilde{u}_h|^2 dx dy + \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(\tilde{u}_h) \} dx dy \right)^{\bar{p}} \right) \right] + \tilde{\mathbb{E}} \left[ \liminf_{h \searrow 0} \int_0^{T_{\max}} \int_{\mathcal{O}} |\tilde{J}_h^x|^2 + |\tilde{J}_h^y|^2 dx dy dt \right] \leq C(\bar{p}, u^0, T_{\max}). \quad (5.105)$$

Similarly as in the proof of Theorem 3.2 in [28], we may use the lower semi-continuity in appropriate topologies to find that

$$\mathbb{E} \left[ \sup_{t \in [0, T_{\max}]} \left( \frac{1}{2} \int_{\mathcal{O}} |\nabla \tilde{u}_h|^2 dx dy \right)^{\bar{p}} + \int_0^{T_{\max}} \int_{\mathcal{O}} |\tilde{J}_h^x|^2 + |\tilde{J}_h^y|^2 dx dy dt \right] \leq C(\bar{p}, u^0, T_{\max}). \quad (5.106)$$

So let us focus on the remaining term  $\int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(\tilde{u}_h) \} dx dy$ .

**Step 1:** There is a positive constant  $C$  such that

$$\int_{\mathcal{O}} F(\tilde{u}_h) dx dy \leq \int_{\mathcal{O}} \mathcal{I}_h^{xy} \{ F(\tilde{u}_h) \} dx dy + C. \quad (5.107)$$

Indeed, with the notation of Assumption **(P)**, we find  $F$  to be convex for  $s \in [0, \hat{u}]$  where  $\hat{u}$  is given by

$$\hat{u} = \sqrt[p+2]{\frac{\tilde{c}_1}{\tilde{c}_2}} \quad (5.108)$$

with the constants  $\tilde{c}_1$  and  $\tilde{c}_2$  introduced in **(P)**. Introducing  $\hat{\delta} := \hat{u}/C_{\text{osc}}$  with  $C_{\text{osc}}$  being the constant in (3.10) and using the definition for  $S_{\hat{\delta}}^{\mathcal{Q}_h}$  from (5.21d), we find that

$$\max_{(x,y) \in \mathcal{O} \setminus S_{\hat{\delta}}^{\mathcal{Q}_h}} \tilde{u}_h \leq \hat{u}. \quad (5.109)$$

Therefore, we have

$$F(\tilde{u}_h) \leq \mathcal{I}_h^{xy} \{ F(\tilde{u}_h) \} \quad \text{on } \mathcal{O} \setminus S_{\hat{\delta}}^{\mathcal{Q}_h} \quad (5.110)$$

due to the convexity of  $F$  on  $(0, \hat{u})$ .

Using **(P)** once more, we find

$$0 \leq F(\tilde{u}_h) \leq C(\hat{\delta}^{-p} + 1) \quad \text{on } S_{\hat{\delta}}^{\mathcal{Q}_h}. \quad (5.111)$$

Combining (5.110) and (5.111), (5.107) is established.

**Step 2:** Estimate on  $\tilde{\mathbb{E}} \left[ \text{ess sup}_{t \in [0, T_{\max}]} \left( \int_{\mathcal{O}} F(\tilde{u}) dx dy \right)^{\bar{p}} \right]$ .

From (5.22a) and (5.22b) together with Sobolev's embedding result, we infer  $\tilde{\mathbb{P}}$ -a.s. that for almost all  $t \in [0, T_{\max}]$

$$\tilde{u}_h(\tilde{\omega}, t, \cdot) \rightarrow \tilde{u}(\tilde{\omega}, t, \cdot) \quad \text{in } C^\gamma(\overline{\mathcal{O}}). \quad (5.112)$$

Hence,  $F(\tilde{u}_h(\tilde{\omega}, t, \cdot)) \rightarrow F(\tilde{u}(\tilde{\omega}, t, \cdot))$  pointwise which implies together with Fatou's lemma

$$\operatorname{ess\,sup}_{t \in [0, T_{\max}]} \int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, t, \cdot)) \, dx \, dy \leq \liminf_{h \searrow 0} \sup_{t \in [0, T_{\max}]} \int_{\mathcal{O}} F(\tilde{u}_h(\tilde{\omega}, t, \cdot)) \, dx \, dy. \quad (5.113)$$

As both sides of inequality (5.113) are non-negative, we can take the  $\bar{p}$ -th power on both sides. Taking the expectation concludes this step.

**Step 3:** Estimate on  $\tilde{\mathbb{E}} \left[ \sup_{t \in [0, T_{\max}]} \left( \int_{\mathcal{O}} F(\tilde{u}) \, dx \, dy \right)^{\bar{p}} \right]$ .

We fix  $t_0 \in [0, T_{\max}]$  arbitrarily and choose  $\tilde{\omega} \in \tilde{\Omega}$  such that  $\tilde{u}(\tilde{\omega}, \cdot, \cdot) \in C([0, T_{\max}]; L^q(\mathcal{O}))$  and  $M(\tilde{\omega}) := \operatorname{ess\,sup}_{t \in [0, T_{\max}]} \int_{\mathcal{O}} F(\tilde{u}) \, dx \, dy < \infty$ . The latter limitations are satisfied by almost all  $\tilde{\omega} \in \tilde{\Omega}$  due to Proposition 5.3 and Step 2.

We will now show that  $\tilde{u}(\tilde{\omega}, t_0, \cdot)$  vanishes only on a set of measure zero. Therefore, we take a sequence  $(s_n)_{n \in \mathbb{N}}$  in  $[0, T_{\max}]$  which satisfies  $s_n \rightarrow t_0$  for  $n \nearrow \infty$ ,

$$\int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, s_n, x, y)) \, dx \, dy \leq M(\tilde{\omega}), \quad (5.114)$$

and  $\tilde{u}(\tilde{\omega}, s_n, \cdot) \rightarrow \tilde{u}(\tilde{\omega}, t_0, \cdot)$  pointwise almost everywhere. Assuming that the set  $\mathcal{N}$ , where  $\tilde{u}(\tilde{\omega}, t_0, \cdot)$  vanishes, has positive measure, we obtain by Egorov's theorem the existence of a set  $\mathcal{N}^\delta \subset \mathcal{N} \subset \mathcal{O}$  such that  $\mu(\mathcal{N}^\delta) > (1 - \delta)\mu(\mathcal{N})$  for  $\delta \in (0, 1)$  such that  $\tilde{u}(\tilde{\omega}, s_n, \cdot) \rightarrow \tilde{u}(\tilde{\omega}, t_0, \cdot)$  uniformly in  $\mathcal{N}^\delta$ . This provides

$$\int_{\mathcal{N}^\delta} F(\tilde{u}(\tilde{\omega}, s_n, x, y)) \, dx \, dy \xrightarrow{n \rightarrow \infty} +\infty. \quad (5.115)$$

However, at the same time we have

$$\int_{\mathcal{N}^\delta} F(\tilde{u}(\tilde{\omega}, s_n, x, y)) \, dx \, dy \leq \int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, s_n, x, y)) \, dx \, dy \leq M(\tilde{\omega}). \quad (5.116)$$

This contradiction provides  $\mu(\mathcal{N}) = 0$  and  $F(\tilde{u}(\tilde{\omega}, s_n, \cdot)) \rightarrow F(\tilde{u}(\tilde{\omega}, t_0, \cdot))$  pointwise almost everywhere. Hence, by applying Fatou's lemma, we obtain

$$\begin{aligned} \int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, t_0, x, y)) \, dx \, dy &\leq \liminf_{n \rightarrow \infty} \int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, s_n, x, y)) \, dx \, dy \\ &\leq M(\tilde{\omega}) = \operatorname{ess\,sup}_{t \in [0, T_{\max}]} \int_{\mathcal{O}} F(\tilde{u}(\tilde{\omega}, t, x, y)) \, dx \, dy. \end{aligned} \quad (5.117)$$

Again, taking the  $\bar{p}$ -th power and the expectation concludes this step.

Together with (5.104) and (5.107), the energy inequality (3.9) is established. Finally, we may combine (5.100a) and (3.9) to deduce the positivity properties of  $\tilde{u}$  claimed in the theorem.  $\square$

**Remark 5.18.** *As the estimate (3.9) is only of a qualitative character we did not strive for an optimal result. In fact, it is — being based on (5.111) — a rather coarse estimate. If more information is available on  $F$ , for instance number and height of local maxima, much better estimates are available — based on appropriate convexity arguments.*

## 6. CONCLUSION

We have proven the existence of martingale solutions to stochastic thin-film equations with conservative linear multiplicative noise in two space dimensions. As our result covers driving noise both in the Itô- and in the Stratonovich-sense, we expect it to be a starting point to construct solutions for solely surface tension driven thin-film evolution (i.e.  $F \equiv 0$ ) subject to compactly supported initial data.<sup>1</sup> This raises questions about the impact of noise on the evolution of the solution's support ("finite speed of propagation" and "waiting time phenomena"). It is well-known in the (deterministic) theory of thin-film equations that analytical concepts in two space dimensions carry over to higher dimensions while the argumentation in dimension  $d = 1$  takes advantage of the Sobolev embedding  $H^1 \hookrightarrow C^{1/2}$  and is therefore much less involved. Hence, a generalization to  $d = 3$  (with the perspective of applications to models for phase separation) is feasible as well. Finally, the implementation of numerical schemes related to the finite element approach presented here will provide further insight into the impact of noise on thin-film evolution.

## APPENDIX A. AUXILIARY RESULTS

**Lemma A.1.** *Let  $\mathcal{Q}_h$  satisfy **(S)** and let  $I$  be the identity operator. Furthermore, let  $p \in [1, \infty)$ ,  $q, r \in [1, \infty]$ ,  $q^* := \frac{q}{q-1}$ , and  $r^* := \frac{r}{r-1}$ . Then the estimates*

$$\|(I - \mathcal{I}_h^x)\{f_h^x g_h^x\}\|_{L^p(\mathcal{O}^x)} \leq Ch_x^2 \|\partial_x f_h^x\|_{L^{pq}(\mathcal{O}^x)} \|\partial_x g_h^x\|_{L^{pq^*}(\mathcal{O}^x)} \quad (\text{A.1a})$$

$$\|\partial_x(I - \mathcal{I}_h^x)\{f_h^x g_h^x\}\|_{L^p(\mathcal{O}^x)} \leq Ch_x \|\partial_x f_h^x\|_{L^{pq}(\mathcal{O}^x)} \|\partial_x g_h^x\|_{L^{pq^*}(\mathcal{O}^x)} \quad (\text{A.1b})$$

$$\|(I - \mathcal{I}_h^y)\{f_h^y g_h^y\}\|_{L^p(\mathcal{O}^y)} \leq Ch_y^2 \|\partial_y f_h^y\|_{L^{pq}(\mathcal{O}^y)} \|\partial_y g_h^y\|_{L^{pq^*}(\mathcal{O}^y)} \quad (\text{A.1c})$$

$$\|\partial_y(I - \mathcal{I}_h^y)\{f_h^y g_h^y\}\|_{L^p(\mathcal{O}^y)} \leq Ch_y \|\partial_y f_h^y\|_{L^{pq}(\mathcal{O}^y)} \|\partial_y g_h^y\|_{L^{pq^*}(\mathcal{O}^y)} \quad (\text{A.1d})$$

hold true for all  $f_h^x, g_h^x \in U_h^x$  and  $f_h^y, g_h^y \in U_h^y$ . In addition the estimates

$$\begin{aligned} \|(I - \mathcal{I}_h^{xy})\{f_h g_h\}\|_{L^p(\mathcal{O})} &\leq Ch_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})} \\ &\quad + Ch_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})}, \end{aligned} \quad (\text{A.1e})$$

$$\|(I - \mathcal{I}_h^x)\{f_h g_h\}\|_{L^p(\mathcal{O})} \leq Ch_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})}, \quad (\text{A.1f})$$

$$\|\mathcal{I}_h^y\{(I - \mathcal{I}_h^x)\{f_h g_h\}\}\|_{L^p(\mathcal{O})} \leq Ch_x^2 \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})}, \quad (\text{A.1g})$$

$$\|\partial_x(I - \mathcal{I}_h^x)\{f_h g_h\}\|_{L^p(\mathcal{O})} \leq Ch_x \|\partial_x f_h\|_{L^{pq}(\mathcal{O})} \|\partial_x g_h\|_{L^{pq^*}(\mathcal{O})}, \quad (\text{A.1h})$$

$$\|(I - \mathcal{I}_h^y)\{f_h g_h\}\|_{L^p(\mathcal{O})} \leq Ch_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})}, \quad (\text{A.1i})$$

$$\|\mathcal{I}_h^x\{(I - \mathcal{I}_h^y)\{f_h g_h\}\}\|_{L^p(\mathcal{O})} \leq Ch_y^2 \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})}, \quad (\text{A.1j})$$

$$\|\partial_y(I - \mathcal{I}_h^y)\{f_h g_h\}\|_{L^p(\mathcal{O})} \leq Ch_y \|\partial_y f_h\|_{L^{pr}(\mathcal{O})} \|\partial_y g_h\|_{L^{pr^*}(\mathcal{O})} \quad (\text{A.1k})$$

hold true for all  $f_h, g_h \in U_h$ .

*Proof.* The estimates (A.1a)–(A.1d) are proven in Lemma 2.1 in [59]. In order to prove (A.1e), we reduce the problem to the one dimensional setting using

$$\|(I - \mathcal{I}_h^{xy})\{f_h g_h\}\|_{L^p(\mathcal{O})}^p \leq \|(I - \mathcal{I}_h^x)\{f_h g_h\}\|_{L^p(\mathcal{O})}^p + \|\mathcal{I}_h^y\{(I - \mathcal{I}_h^x)\{f_h g_h\}\}\|_{L^p(\mathcal{O})}^p =: A + B \quad (\text{A.2})$$

<sup>1</sup>In the case of Itô-noise, no formal a priori-estimates are known for the case of compactly supported initial data.

and apply (A.1a)–(A.1d). This provides

$$A \leq Ch_y^{2p} \int_{\mathcal{O}^x} \|\partial_y f_h\|_{L^{pq}(\mathcal{O}^y)}^p \|\partial_y g_h\|_{L^{pq^*}(\mathcal{O}^y)}^p dx \leq Ch_y^{2p} \|\partial_y f_h\|_{L^{pq}(\mathcal{O})}^p \|\partial_y g_h\|_{L^{pq^*}(\mathcal{O})}^p. \quad (\text{A.3})$$

Using Jensen's inequality, (2.6), and Hölder's inequality, we obtain for  $r \in (1, \infty)$

$$\begin{aligned} B &\leq \int_{\mathcal{O}^y} \mathcal{I}_h^y \left\{ \int_{\mathcal{O}^x} |(I - \mathcal{I}_h^x)\{f_h g_h\}|^p dx \right\} dy \leq Ch_x^{2p} \int_{\mathcal{O}^y} \mathcal{I}_h^y \left\{ \|\partial_x f_h\|_{L^{pr}(\mathcal{O}^x)}^p \|\partial_x g_h\|_{L^{pr^*}(\mathcal{O}^x)}^p \right\} dy \\ &\leq Ch_x^{2p} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x f_h|^{pr} \} dx dy \right)^{1/r} \left( \int_{\mathcal{O}} \mathcal{I}_h^y \{ |\partial_x g_h|^{pr^*} \} dx dy \right)^{1/r^*} \\ &\leq Ch_x^{2p} \|\partial_x f_h\|_{L^{pr}(\mathcal{O})}^p \|\partial_x g_h\|_{L^{pr^*}(\mathcal{O})}^p. \end{aligned} \quad (\text{A.4})$$

The corresponding estimate for  $r \in \{1, \infty\}$  is straightforward. The estimates (A.1f)–(A.1k) can be derived from (A.1a)–(A.1d) in a similar manner.  $\square$

**Lemma A.2.** *Let Assumption (S) hold true and let the operator  $A_h : U_h \rightarrow U_h \cap H_*^1(\mathcal{O})$  be defined by*

$$\int_{\mathcal{O}} (A_h \phi_h) \psi_h dx dy = \int_{\mathcal{O}} \nabla \phi_h \cdot \nabla \psi_h dx dy \quad \forall \psi_h \in U_h. \quad (\text{A.5})$$

Then there exists a constant  $C < 0$ , such that for all  $\phi_h \in U_h$  the estimates

$$\|\phi_h\|_{L^\infty(\mathcal{O})} \leq C \|A_h \phi_h\|_{L^2(\mathcal{O})}^\kappa \|\phi_h\|_{H^1(\mathcal{O})}^{1-\kappa} + C \|\phi_h\|_{H^1(\mathcal{O})}, \quad (\text{A.6a})$$

$$\|\phi_h\|_{W^{1,p}(\mathcal{O})} \leq C \|A_h \phi_h\|_{L^2(\mathcal{O})}^\nu \|\phi_h\|_{H^1(\mathcal{O})}^{1-\nu} + C \|\phi_h\|_{H^1(\mathcal{O})}, \quad (\text{A.6b})$$

hold true with  $\kappa = \frac{1}{2}$ ,  $\nu = \frac{3p-6}{2p}$  and  $p \in [2, 6]$  for  $d = 3$ , and  $\kappa = \frac{1}{4}$ ,  $\nu = \frac{p-2}{p}$  and  $p \in [2, \infty)$  for  $d = 2$ .

*Proof.* A similar result for continuous, piecewise linear finite element functions defined on a simplicial triangulation has been proven in Lemma A.1 in [59]. This proof relies solely on standard error estimates for second order problems (see e.g. Chapter 3 in [13]), standard inverse estimates for finite element functions (see e.g. Theorem 4.5.11 in [10]), and the properties of the Clément interpolation operator (see e.g. Lemma 1.127 in [23]). As the first two results remain valid for quadrilateral finite element spaces satisfying (S) and the Clément interpolation operator, which was originally only defined for simplicial elements, can be replaced by the more general operator proposed in [4] that satisfies the same error estimates as the Clément interpolation operator, we refer the reader to [59].  $\square$

**Lemma A.3.** *Let Assumption (S) hold true and let the operator  $A_h : U_h \rightarrow U_h \cap H_*^1(\mathcal{O})$  be defined by*

$$\int_{\mathcal{O}} (A_h \phi_h) \psi_h dx dy = \int_{\mathcal{O}} \nabla \phi_h \cdot \nabla \psi_h dx dy \quad \forall \psi_h \in U_h. \quad (\text{A.7})$$

Then there exists a constant  $C < 0$ , such that for all  $\phi_h \in U_h$  the estimate

$$\|A_h \phi_h\|_{L^2(\mathcal{O})} \leq C \|\Delta_h \phi_h\|_{L^2(\mathcal{O})}, \quad (\text{A.8})$$

where  $\Delta_h$  denotes the discrete Laplacian defined in (2.7).

*Proof.* Noting that the definitions of  $\Delta_h^x$  and  $\Delta_h^y$  in (2.7) imply

$$\begin{aligned} - \int_{\mathcal{O}^x} \mathcal{I}_h^x \{ \Delta_h^x \phi_h \psi_h \} dx &= \int_{\mathcal{O}^x} \partial_x \phi_h \partial_x \psi_h dx && \text{pointwise for all } y \in \mathcal{O}^y, \\ - \int_{\mathcal{O}^y} \mathcal{I}_h^y \{ \Delta_h^y \phi_h \psi_h \} dy &= \int_{\mathcal{O}^y} \partial_y \phi_h \partial_y \psi_h dy && \text{pointwise for all } x \in \mathcal{O}^x, \end{aligned} \quad (\text{A.9})$$

we compute

$$\begin{aligned} \|A_h \phi_h\|_{L^2(\mathcal{O})}^2 &= \int_{\mathcal{O}} \partial_x \phi_h \partial_x A_h \phi_h dx dy + \int_{\mathcal{O}} \partial_y \phi_h \partial_y A_h \phi_h dx dy \\ &= \int_{\mathcal{O}} \mathcal{I}_h^x \{ (-\Delta_h^x \phi_h) A_h \phi_h \} dx dy + \int_{\mathcal{O}} \mathcal{I}_h^y \{ (-\Delta_h^y \phi_h) A_h \phi_h \} dx dy \\ &\leq C \|\Delta_h^x \phi_h\|_{L^2(\mathcal{O})} \|A_h \phi_h\|_{L^2(\mathcal{O})} + C \|\Delta_h^y \phi_h\|_{L^2(\mathcal{O})} \|A_h \phi_h\|_{L^2(\mathcal{O})}, \end{aligned} \quad (\text{A.10})$$

which concludes the proof.  $\square$

The following corollary is a direct consequence of Lemma A.2 and Lemma A.3.

**Corollary A.4.** *Let Assumption (S) hold true and let  $\Delta_h$  be the discrete Laplacian defined in (2.7). Then there exists a constant  $C < \infty$ , such that for all  $\phi_h \in U_h$  the estimates*

$$\|\phi_h\|_{L^\infty(\mathcal{O})} \leq C \|\Delta_h \phi_h\|_{L^2(\mathcal{O})}^\kappa \|\phi_h\|_{H^1(\mathcal{O})}^{1-\kappa} + C \|\phi_h\|_{H^1(\mathcal{O})}, \quad (\text{A.11a})$$

$$\|\phi_h\|_{W^{1,p}(\mathcal{O})} \leq C \|\Delta_h \phi_h\|_{L^2(\mathcal{O})}^\nu \|\phi_h\|_{H^1(\mathcal{O})}^{1-\nu} + C \|\phi_h\|_{H^1(\mathcal{O})}, \quad (\text{A.11b})$$

hold true with  $\kappa = \frac{1}{2}$ ,  $\nu = \frac{3p-6}{2p}$  and  $p \in [2, 6]$  for  $d = 3$ , and  $\kappa = \frac{1}{4}$ ,  $\nu = \frac{p-2}{p}$  and  $p \in [2, \infty)$  for  $d = 2$ .

**Lemma A.5.** *Let  $G : \mathbb{R} \supset I \rightarrow \mathbb{R}$  be of class  $W^{1,\infty}$ , let a partition  $\mathcal{Q}_h$  of a domain  $\mathcal{O}$  satisfying (S) be given. Furthermore, let  $U_h$  be the corresponding space defined in (2.2c). Then the estimate*

$$\|\mathcal{I}_h^{xy} \{ G(u_h) \} - G(u_h)\|_{L^2(\mathcal{O})}^2 \leq C \|G\|_{W^{1,\infty}}^2 h^2 \int_{\mathcal{O}} |\nabla u_h|^2 dx dy \quad (\text{A.12})$$

holds true for arbitrary  $u_h \in U_h$ .

*Proof.* We will prove that the claim holds true on every  $Q \in \mathcal{Q}_h$ . After an appropriate translation, we may assume that  $Q$  is given by  $Q = \text{co}\{(0, 0), (h_x, 0), (h_x, h_y), (0, h_y)\}$ . For arbitrary  $(x, y) \in Q$ , we have

$$\begin{aligned} G(u_h(x, y)) &= \frac{1}{2}G(u_h(x, 0)) + \frac{1}{2}(\partial_y G(u_h))(x, \xi_1^y) \cdot y \\ &\quad + \frac{1}{2}G(u_h(x, h_y)) + \frac{1}{2}(\partial_y G(u_h))(x, \xi_2^y) \cdot (y - h_y) \\ &= \frac{1}{2}G(u_h(0, 0)) + \frac{1}{2}(\partial_x G(u_h))(\xi_1^x, 0) \cdot x + \frac{1}{2}(\partial_y G(u_h))(x, \xi_1^y) \cdot y \\ &\quad + \frac{1}{2}G(u_h(h_x, h_y)) + \frac{1}{2}(\partial_x G(u_h))(\xi_2^x, h_y) \cdot (x - h_x) \\ &\quad + \frac{1}{2}(\partial_y G(u_h))(x, \xi_2^y) \cdot (y - h_y) \end{aligned} \quad (\text{A.13})$$

with  $\xi_1^x, \xi_2^x \in [0, h_x]$  and  $\xi_1^y, \xi_2^y \in [0, h_y]$ . Similarly, we have

$$\begin{aligned} (\mathcal{I}_h^{xy}\{G(u_h)\})(x, y) &= \frac{1}{2}(\mathcal{I}_h^{xy}\{G(u_h)\})(0, 0) + \frac{1}{2}(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_1^x, 0) \cdot x \\ &\quad + \frac{1}{2}(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x, \zeta_1^y) \cdot y + \frac{1}{2}(\mathcal{I}_h^{xy}\{G(u_h)\})(h_x, h_y) \\ &\quad + \frac{1}{2}(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_2^x, h_y) \cdot (x - h_x) \\ &\quad + \frac{1}{2}(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x, \zeta_2^y) \cdot (y - h_y) \end{aligned} \quad (\text{A.14})$$

with  $\zeta_1^x, \zeta_2^x \in [0, h_x]$  and  $\zeta_1^y, \zeta_2^y \in [0, h_y]$ . Therefore, we have

$$\begin{aligned} &|G(u_h(x, y)) - (\mathcal{I}_h^{xy}\{G(u_h)\})(x, y)|^2 \\ &\leq Ch_x^2 |(\partial_x G(u_h))(\xi_1^x, 0)|^2 + Ch_y^2 |(\partial_y G(u_h))(x, \xi_1^y)|^2 + Ch_x^2 |(\partial_x G(u_h))(\xi_2^x, h_y)|^2 \\ &\quad + Ch_y^2 |(\partial_y G(u_h))(x, \xi_2^y)|^2 + Ch_x^2 |(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_1^x, 0)|^2 \\ &\quad + Ch_y^2 |(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x, \zeta_1^y)|^2 + Ch_x^2 |(\partial_x \mathcal{I}_h^{xy}\{G(u_h)\})(\zeta_2^x, h_y)|^2 \\ &\quad + Ch_y^2 |(\partial_y \mathcal{I}_h^{xy}\{G(u_h)\})(x, \zeta_2^y)|^2 \\ &\leq Ch^2 \|G\|_{W^{1,\infty}}^2 (|\partial_x u_h(\cdot, 0)|^2 + |\partial_x u_h(\cdot, h_y)|^2 + |\partial_y u_h(x, \cdot)|^2) \\ &\quad + Ch^2 \|G\|_{W^{1,\infty}}^2 (|\partial_x u_h(\cdot, 0)|^2 + |\partial_x u_h(\cdot, h_y)|^2 + \mathcal{I}_h^x \{|\partial_y u_h(x, \cdot)|^2\}). \end{aligned} \quad (\text{A.15})$$

Integrating over  $Q$  and using the norm equivalence stated in (2.6) provides the result.  $\square$

**Lemma A.6.** *Let Assumption (S) hold true. Then the identities*

$$\begin{aligned} - \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{b_h^x(x) \partial_x^{-h_x} c_h^x(x)\} dx &= \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{\partial_x^{-h_x} b_h^x(x) c_h^x(x - h_x)\} dx \\ &\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x) \mathcal{I}_h^x \{b_h^x(x) c_h^x(x)\} dx \end{aligned} \quad (\text{A.16a})$$

$$\begin{aligned} \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{b_h^x(x) \Delta_h^x c_h^x(x)\} dx &= \int_{\mathcal{O}^x} \Delta_h^x a_h^x(x) \mathcal{I}_h^x \{b_h^x(x - h_x) c_h^x(x)\} dx \\ &\quad + \int_{\mathcal{O}^x} a_h^x(x + h_x) \mathcal{I}_h^x \{\Delta_h^x b_h^x(x) c_h^x(x)\} dx \\ &\quad + 2 \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x) \mathcal{I}_h^x \{\partial_x^{-h_x} b_h^x(x) c_h^x(x)\} dx \end{aligned} \quad (\text{A.16b})$$

$$\begin{aligned} - \int_{\mathcal{O}^y} a_h^y(y) \mathcal{I}_h^y \{b_h^y(y) \partial_y^{-h_y} c_h^y(y)\} dy &= \int_{\mathcal{O}^y} a_h^y(y) \mathcal{I}_h^y \{\partial_y^{-h_y} b_h^y(y) c_h^y(y - h_y)\} dy \\ &\quad + \int_{\mathcal{O}^y} \partial_y^{+h_y} a_h^y(y) \mathcal{I}_h^y \{b_h^y(y) c_h^y(y)\} dy \end{aligned} \quad (\text{A.16c})$$

$$\begin{aligned} \int_{\mathcal{O}^y} a_h^y(y) \mathcal{I}_h^y \{b_h^y(y) \Delta_h^y c_h^y(y)\} dy &= \int_{\mathcal{O}^y} \Delta_h^y a_h^y(y) \mathcal{I}_h^y \{b_h^y(y - h_y) c_h^y(y)\} dy \\ &\quad + \int_{\mathcal{O}^y} a_h^y(y + h_y) \mathcal{I}_h^y \{\Delta_h^y b_h^y(y) c_h^y(y)\} dy \\ &\quad + 2 \int_{\mathcal{O}^y} \partial_y^{+h_y} a_h^y(y) \mathcal{I}_h^y \{\partial_y^{-h_y} b_h^y(y) c_h^y(y)\} dy \end{aligned} \quad (\text{A.16d})$$

hold true for periodic functions  $a_h^x \in C_{\text{per}, \mathcal{T}_h^x}$ ,  $b_h^x, c_h^x \in U_h^x$ ,  $a_h^y \in C_{\text{per}, \mathcal{T}_h^y}$ , and  $b_h^y, c_h^y \in U_h^y$ .

*Proof.* To show the second identity in (A.16a), we use the periodicity of the considered functions and compute

$$\begin{aligned}
& -h_x \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ b_h^x(x) \partial_x^{-h_x} c_h^x(x) \} dx \\
&= \int_{\mathcal{O}^x} a_h^x(x+h_x) \mathcal{I}_h^x \{ b_h^x(x+h_x) c_h^x(x) \} dx - \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ b_h^x(x) c_h^x(x) \} dx \\
&= \int_{\mathcal{O}^x} a_h^x(x+h_x) \mathcal{I}_h^x \{ (b_h^x(x+h_x) - b_h^x(x)) c_h^x(x) \} dx \\
&\quad + \int_{\mathcal{O}^x} (a_h^x(x+h_x) - a_h^x(x)) \mathcal{I}_h^x \{ b_h^x(x) c_h^x(x) \} dx \\
&= \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ (b_h^x(x) - b_h^x(x-h_x)) c_h^x(x-h_x) \} dx \\
&\quad + \int_{\mathcal{O}^x} (a_h^x(x+h_x) - a_h^x(x)) \mathcal{I}_h^x \{ b_h^x(x) c_h^x(x) \} dx .
\end{aligned} \tag{A.17}$$

We will now use (A.16a) to prove (A.16b).

$$\begin{aligned}
& \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ b_h^x(x) \partial_x^{-h_x} \partial_x^{+h_x} c_h^x(x) \} dx \\
&= - \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ \partial_x^{-h_x} b_h^x(x) \partial_x^{+h_x} c_h^x(x-h_x) \} dx - \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x) \mathcal{I}_h^x \{ b_h^x(x) \partial_x^{+h_x} c_h^x(x) \} dx \\
&= - \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ \partial_x^{-h_x} b_h^x(x) \partial_x^{-h_x} c_h^x(x) \} dx \\
&\quad - \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x-h_x) \mathcal{I}_h^x \{ b_h^x(x-h_x) \partial_x^{-h_x} c_h^x(x) \} dx \\
&= \int_{\mathcal{O}^x} a_h^x(x) \mathcal{I}_h^x \{ \partial_x^{-h_x} \partial_x^{-h_x} b_h^x(x) c_h^x(x-h_x) \} dx + \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x) \mathcal{I}_h^x \{ \partial_x^{-h_x} b_h^x(x) c_h^x(x) \} dx \\
&\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x-h_x) \mathcal{I}_h^x \{ \partial_x^{-h_x} b_h^x(x-h_x) c_h^x(x-h_x) \} dx \\
&\quad + \int_{\mathcal{O}^x} \partial_x^{+h_x} \partial_x^{+h_x} a_h^x(x-h_x) \mathcal{I}_h^x \{ b_h^x(x-h_x) c_h^x(x) \} dx \\
&= \int_{\mathcal{O}^x} a_h^x(x+h_x) \mathcal{I}_h^x \{ \Delta_h^x b_h^x(x) c_h^x(x) \} dx + 2 \int_{\mathcal{O}^x} \partial_x^{+h_x} a_h^x(x) \mathcal{I}_h^x \{ \partial_x^{-h_x} b_h^x(x) c_h^x(x) \} dx \\
&\quad + \int_{\mathcal{O}^x} \Delta_h^x a_h^x(x) \mathcal{I}_h^x \{ b_h^x(x-h_x) c_h^x(x) \} dx .
\end{aligned} \tag{A.18}$$

The identities (A.16c) and (A.16d) can be established in a similar manner.  $\square$

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